Application of the repeated quantization method to the problem of making asymptotic solutions of equations with holomorphic coefficients

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***Abstract*— In this work, we derive asymptotics of solutions of ordinary differential equations with holomorphic coefficients in the neighborhood of infinity.**

This problem represents a particular case of the general problem of constructing asymptotics of linear differential equations with irregular singularities, namely the Poincare problem. The case of infinitely distant singular point is an example of irregular singularity and the problem of derivation of asymptotics of its solutions is reduced to the problem of constructing asymptotics of solutions in the neighborhood of zero of linear differential equations with the cusp-type singularity of the second order. If the principal symbol of differential operator has simple roots, then asymptotics of solution of equation in the neighborhood of an irregular singular point can be represented as a classic non-Fuchs asymptotics, which is a familiar fact. In the case of multiple roots, the method of repeated quantization is used. The method is based on the Laplace-Borel transform. Using repeated quantization in this paper we solve the problem of derivation of asymptotics of solutions in the neighborhood of infinity for a model problem whose singularity index has a special form. The derived asymptotics of solutions differ from the classic non-Fuchs asymptotics and represent their generalizations. The method of solution of this model problem in its essential part is transferred to the general case. Thus, this work is one of steps in solving Poincare problem.

***Keywords*—differential equations with cuspidal singularitus, Laplas-Borel transformation, resurgent function, principle operator symbol, asymptotic expansion.**

I. Introduction

The work aims to analyze methods for constructing asymptotic solutions in ordinary differential equations with holomorphic coefficients with degeneracies. Namely, we study ordinary differential equations with holomorphic coefficients

$$\begin{aligned} & b_n(r)\left(\frac{d}{dr}\right)^n u(r)+b_{n-1}(r)\left(\frac{d}{dr}\right)^{n-1} u(r)+\dots \\ & +b_i(r)\left(\frac{d}{dr}\right)^i u(r)+\dots+b_0(r)u(r)=0 \end{aligned} \quad (1)$$

here, $b_i(r)$ are holomorphic functions.

If the coefficient of the highest derivative $b_n(r)$ vanishes at some point, without loss of generality, it can be assumed that this point is $r=0$, then the equation (1), generally speaking, has a singularity at zero. In this case, zero can be a regular or irregular singular point. The problem of representing asymptotic solutions to an equation with holomorphic coefficients near an irregular singular point was first formulated by H. Poincaré in [1], [2]. In these papers, it was first shown that the solution of an equation with holomorphic coefficients near an irregular singular point in some cases can be decomposed into an asymptotic series. One of the possible methods for summing the asymptotic divergent series - using integral transforms - was also formulated by Poincaré in [2]. As the integral transform, Poincaré used the Laplace transform, but it is applicable only in some special cases. In this paper, the Laplace-Borel transform, which was introduced by Ecalle in [3] and is the basis of resurgent analysis, will be used to sum the corresponding asymptotic series.

Thome's work was one of the first papers, considering the problem of making asymptotic solutions in the vicinity an irregular singular point [4]. An equation with holomorphic coefficients is considered.

$$\begin{aligned} & \left(\frac{d}{dx}\right)^n u(x)+a_{n-1}(x)\left(\frac{d}{dx}\right)^{n-1} u(x)+\dots \\ & +a_i(x)\left(\frac{d}{dx}\right)^i u(x)+\dots+a_0(x)u(x)=0 \end{aligned}, \quad (2)$$

here, the coefficients $a_i(x)$ are regular at infinity, this means that there is such an exterior of the circle $|x|>a$ that the functions $a_i(x), i=0,1,\dots,n-1$ decompose in it into convergent power series $a_i(x)=\sum_{j=0}^{\infty}\frac{b_i^j}{x^j}$. Our study aims at making asymptotic solutions of the equation (2) in the vicinity of infinity.

II. Main Results

Let's note that the equation (1) can be reduced to an equation, looking like this

$$\hat{H}u = H\left(r,-r^k\frac{d}{dr}\right)u=0, \quad (3)$$

where $\hat{H}$ is a differential operator with holomorphic coefficients

$$H(r,p)=\sum_{i=0}^{n}a_i(r)p^i .$$

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Here, $a_i(r)$ are holomorphic functions, and $a_n(0)\neq 0$. In [4], it was shown that one can find a minimum integer nonnegative *k* and a formula for calculating this minimum value *k* was obtained. The equation (3) is called the equation with the cuspidal degeneration of the k-th order.

By replacement of $x=\dfrac{1}{r}$, the problem (2) also reduces to the equation (3) for the case, when $k=2$. In other words, the problem of constructing asymptotic solutions of the equation (2) at infinity requires to study the equation with a second-order cuspidal degeneracy. In this case,

$$H(r,p)=p^n+\sum_{i=0}^{n-1}a_i(r)p^i \tag{4}$$

In the beginning, we will consider the case, when the principal symbol of a differential operator $H(0,p)$, has one root; without loss of generality, we assume that this root is at zero. It follows that in this case $a_i(x)=\sum_{j=1}^{\infty}\dfrac{b_i^j}{x^j}$.

Let's write the equation (2) as

$$\begin{gathered}\left(-r^2\frac{d}{dr}\right)^n u+a_0r^{m_k}\left(-r^2\frac{d}{dr}\right)^k u+a_1r^{m_k+1}\left(-r^2\frac{d}{dr}\right)^{k-1}u+\\ +a_2r^{m_k+2}\left(-r^2\frac{d}{dr}\right)^{k-2}u+\ldots+a_{k+1}r^{m_k+k}u+\\ \ldots+\sum_{j=1}^{h}r^j\sum_{i=h_j}^{n-1}a_j^i\left(-r^2\frac{d}{dr}\right)^i u+r^{h+1}\sum_{i=0}^{n-1}a^i(r)\left(-r^2\frac{d}{dr}\right)^i u=0\end{gathered} \tag{5}$$

Here, $h_j+j>m_k+k$. Let's $h=m_k+k$ call the singularity index; in other words, members of $a_j^ir^j\left(r^2\dfrac{d}{dr}\right)^i$ provided that $j+i>h$ are the minor members. Let's divide them into two types. To the first type, let's assign members, for which $h\geq j$, and to the second type – $h<j$. In this article we will consider a special case of this problem. We will consider the case, when $m_k=1$; i.e., the singularity index is $1+k$. This equation is a model, and the asymptotics construction at infinity is an important step in solving the problem of constructing asymptotic solutions of the equation (3) in the general case.

The method of constructing the asymptotic solutions of the model problem is largely carried over to the general case. We show that will be fair

THEOREM. The asymptotic solution of the equation (2) *with* $x\to\infty$ *is*

$$\begin{gathered}u(x)\approx\\ \approx\sum_{j=1}^{n-k}\exp\left(\sum_{i=1}^{n-k-1}\alpha_i^j x^{\frac{i}{n-k}}\right)x^{-\frac{\sigma_i}{n-k}}\sum_{l}^{\infty}A_l^j x^{-\frac{l}{n-k}}+\\ +\sum_{j=0}^{k_0}\left(\ln\frac{1}{x}\right)^j x^{\alpha_j}\sum_{i=0}^{\infty}b_i^j x^{-i},\end{gathered} \tag{6}$$

where $\alpha_{n-k-1}^j,\ j=1,\ldots,n-k$ *are polynomial roots* $p^{n-k}+a_0\left(\dfrac{n-k}{n-k-1}\right)^{n-k}$, σ_i,α_i,k_0 *and* $\alpha_i^j,\ j=1,\ldots,n-k-2$ *are some numbers; if* $n\leq h$, *then* $A_l^j=0,\forall i,j$.

This theorem is the main result of this paper; the proof of this theorem is given below.

Proof.

Without loss of generality, let's assume that the equation includes only one term of the first type and one term of the second type, namely

$$\begin{gathered}\left(-r^2\frac{d}{dr}\right)^n u+a_0r\left(-r^2\frac{d}{dr}\right)^k u+a_1r^2\left(-r^2\frac{d}{dr}\right)^{k-1}u+\\ +a_2r^3\left(-r^2\frac{d}{dr}\right)^{k-2}u+\ldots+a_{i-1}r^i\left(-r^2\frac{d}{dr}\right)^{k-i+1}+\ldots\\ +a_kr^{k+1}u+b_1r^i\left(-r^2\frac{d}{dr}\right)^{k-i+1+\beta_1}u+\\ +b_2r^{k+1+\beta_2}\left(-r^2\frac{d}{dr}\right)^{m_0}u=0\end{gathered} \tag{7}$$

Here, the last two terms belong to the lower members; one of them belongs to the first type, and the second to the second type. We will look for the asymptotic solution of the equation (7) in the vicinity of zero, using the repeated quantization method, see [6].

The proof of the theorem can be divided into several stages. At the first stage, the equation is transformed, using the Laplace-Borel transform, and the singular points of the transformed equation's solution are determined. At the second stage, using the repeated quantization method, asymptotic solutions are made near singular points; then the inverse Laplace-Borel transform from the asymptotics is taken.

Let's recall the definition of the Laplace-Borel transform.

Let's denote $S_{R,\varepsilon}=\{r\,|-\varepsilon<\arg r<\varepsilon,|r|<R\}$ by sector $S_{R,\varepsilon}$. We will seek a solution to the equation (7) in the space $E_k(S_{R,\varepsilon})$ of holomorphic functions in the domain $S_{R,\varepsilon}$ that grow k-exponentially at zero.

By the $E(\tilde{\Omega}_{R,\varepsilon})$, let's denote the space of holomorphic functions of exponential growth in the domain $\tilde{\Omega}_{R,\varepsilon}=\left\{p\,\middle|-\dfrac{\pi}{2}-\varepsilon<\arg p<\dfrac{\pi}{2}+\varepsilon,|p|>R\right\}$; by $E(C)$, the space of entire functions of exponential growth will be denoted.

k-th Laplace-Borel transform of the function $f(r)\in E_k(S_{R,\varepsilon})$ is called a function $B_k:E_k(S_{R,\varepsilon})\to \mathrm{E}(\tilde{\Omega}_{R,\varepsilon})/\mathrm{E(C)}$

$$\hat{f}(p) = B_k f = \int_0^{r_0} e^{-p/r^k} f(r) \frac{dr}{r^{k+1}}.$$

The inverse Laplace-Borel k-transform is defined by the formula

$$B_k^{-1} f = \frac{k}{2\pi i} \int_{\tilde{\gamma}} e^{p/r^k} \hat{f}(p)\, dp$$

The loop $\tilde{\gamma}$ is depicted in Fig. 1, in [7].

Let's apply the Laplace-Borel transform to the equation (7), see [8]. The converted equation (7) will look like this

$$\begin{gathered} p^n \hat{u}(p) - a_0 \int_1^p p^k \hat{u}(p) dp + \\ + a_1 (-1)^2 \int_1^p \int_1^{p_2} p_1^{k-1} \hat{u}(p_1) dp_1 dp_2 + \ldots \\ + a_{i-1} (-1)^i \int_1^p \ldots \int_1^{p_2} p_1^{k-i+1} \hat{u}(p_1) dp_1 \ldots dp_i + \ldots \\ + a_k (-1)^{k+1} \int_1^p \ldots \int_1^{p_2} \hat{u}(p_1) dp_1 \ldots dp_{k+1} + \\ + b_1 (-1)^i \int_1^p \ldots \int_1^{p_2} p_1^{k-i+1+\beta_1} \hat{u}(p_1) dp_1 \ldots dp_i + \\ + b_2 (-1)^{k+\beta_2+1} \int_1^p \ldots \int_1^{p_2} p^{m_0} \hat{u}(p_1) dp_1 \ldots dp_{k+\beta_2+1} = f(p) \end{gathered} \tag{8}$$

Here, f is an arbitrary holomorphic function. Let's consider the case $n < h$. Let's rewrite the equation (8) as

$$\begin{gathered} \hat{u}(p) = \frac{a_0}{p^n} \int_1^p p^k \hat{u}(p) dp - \\ \frac{a_1 (-1)^2}{p^n} \int_1^p \int_1^{p_2} p_1^{k-1} \hat{u}(p_1) dp_1 dp_2 + \ldots \\ - \frac{a_{i-1} (-1)^i}{p^n} \int_1^p \ldots \int_1^{p_2} p_1^{k-i+1} \hat{u}(p_1) dp_1 \ldots dp_i - \ldots \\ - \frac{a_k (-1)^{k+1}}{p^n} \int_1^p \ldots \int_1^{p_2} \hat{u}(p_1) dp_1 \ldots dp_{k+1} - \\ \frac{b_1 (-1)^i}{p^n} \int_1^p \ldots \int_1^{p_2} p_1^{k-i+1+\beta_1} \hat{u}(p_1) dp_1 \ldots dp_i - \\ - \frac{b_2 (-1)^{k+\beta_2+1}}{p^n} \int_1^p \ldots \int_1^{p_2} p^{m_0} \hat{u}(p) dp_1 \ldots dp_{k+\beta_2+1} + \frac{f(p)}{p^n} \end{gathered} \tag{9}$$

Let's apply the method of successive approximations to the equation (9), then, just as it was done in [9], it can be shown that the asymptotic solutions of the equation (9) in the vicinity of the point $p = 0$ are conormal.

Now let $n \geq h$. Let's differentiate the equation *k+1* times; we get the equation

$$\begin{gathered} \left(\frac{d}{dp}\right)^{k+1} p^n \hat{u}(p) - a_0 \left(\frac{d}{dp}\right)^k p^k \hat{u}(p) + \\ + a_1 (-1)^2 \left(\frac{d}{dp}\right)^{k-1} p^{k-1} \hat{u}(p) + \ldots \\ + a_{i-1} (-1)^i \left(\frac{d}{dp}\right)^{k+1-i} p^{k+1-i} \hat{u}(p) + \ldots \\ + a_k \hat{u}(p) + b_1 (-1)^i \left(\frac{d}{dp}\right)^{k+1-i} p^{k-i+1+\beta_1} \hat{u}(p) + \\ + b_2 (-1)^{k+\beta_2+1} \int_1^p \ldots \int_1^{p_2} p_1^{m_0} \hat{u}(p_1) dp_1 \ldots dp_{\beta_2} = \left(\frac{d}{dp}\right)^{k+1} f \end{gathered} \tag{10}$$

Let's note that, when $n = h$, we obtain an equation with a conic degeneracy. As is known, the solution of such an equation has a conormal asymptotics at zero. Let's assume that $n > h$. This is the most difficult case.

Let's multiply the equation (10) by $p^{k(n-k-1)}$. It is easy to show that the equation (10) can be rewritten as

$$\begin{gathered} \left(-\frac{1}{n-k-1} p^{n-k} \frac{d}{dp}\right)^{k+1} \hat{u}(p) + \\ + a_0^1 \left(-\frac{1}{n-k-1} p^{n-k} \frac{d}{dp}\right)^k \hat{u}(p) + \\ + a_1^1 p^{(n-k-1)} \left(-\frac{1}{n-k-1} p^{n-k} \frac{d}{dp}\right)^{k-1} \hat{u}(p) + \\ + a_2^1 p^{2(n-k-1)} \left(-\frac{1}{n-k-1} p^{n-k} \frac{d}{dp}\right)^{k-2} \hat{u}(p) + \ldots \\ + a_k^1 p^{k(n-k-1)} \hat{u}(p) + \\ + \sum_{j=0}^{k-i} b_1^j p^{\beta_1 + (i+j-1)(n-k-1)} \left(-\frac{1}{n-k-1} p^{n-k} \frac{d}{dp}\right)^{k+1-i-j} \hat{u}(p) + \\ + b_2 p^{k(n-k-1)} \int_1^p \ldots \int_1^{p_2} p_1^{m_0} \hat{u}(p_1) dp_1 \ldots dp_{\beta_2} = \\ = p^{k(n-k-1)} \left(\frac{d}{dp}\right)^{k+1} f(p) \end{gathered} \tag{11}$$

Here, $a_0^1 = \frac{a_0}{n-k-1}$, $a_i^1, i = 1, \ldots, k$, $b_1^j, j = 0, \ldots, k-i$ are corresponding numbers. Let's note that the equation (11) differs from an equation of *n-k* order of cuspidal degeneracy; we only have an integral member $p^{k(n-k-1)} \int_1^p \ldots \int_1^{p_2} p_1^{m_0} \hat{u}(p_1) dp_1 \ldots dp_{\beta_2}$. In [6], it was proved that $B\left(\int_{q_0}^p u(p) dp\right) = -\frac{1}{q} \int_{q_0}^q \int_{q_0}^{q_2} \tilde{u}(q_1) dq_1 dq_2$; from this it follows that the Laplace-Borel transform increases the multiplicity of the integral, so the proof of the resurgence of the solution, given in [7], [10] can be transferred without change to the equation (8). Solution of the equation (11) is a resurgent function. It follows that the Laplace-Borel transform can be applied to this equation in the same way as it was done for the equation (7).

To make asymptotic solution of the equation (11) at $p \to 0$, we apply the repeated quantization method [6]. To do this, in the equation (11), we make for the Laplace-Borel $(n-k)$ transform. The main symbol of the differential function on the left side of the equation (11) is $q^{k+1} + \frac{a_0}{n-k-1}q^k = q^k\left(q + \frac{a_0}{n-k-1}\right)$. The principal symbol has two roots: $q = 0$ and $q = -\frac{a_0}{n-k-1}$. Let's introduce the notation $c = -\frac{a_0}{n-k-1}$. We will find the asymptotic solution of the transformed equation (11). First let's find an asymptotic member, corresponding to the root $q = c$. Without loss of generality, let's assume that $a_1^1 = 0$. If this is not the case, this factor can be reset by replacing $\hat{u}_1(p) = p^{\sigma}\hat{u}(p)$.

Let's find the asymptotics of the Laplace-Borel transform of the equation's (11) right side in the vicinity the point $q = c$

$$\begin{aligned} B_{n-k}p^{k(n-k-1)}f &= B_{n-k}p^{k(n-k-1)}\sum_{i=0}^{\infty} b_i p^i = \\ &= a_0 q^{k-1}\ln q + a_1 q^{\frac{k(n-k-1)+1}{n-k-1}-1} + \sum_{i=2}^{\infty} a_i q^{\frac{k(n-k-1)+i}{n-k-1}-1} = \\ &= a_0 q^{k-1}\ln q + \sum_{i=1}^{\infty} a_i q^{k-1+\frac{i}{n-k-1}} \end{aligned} \qquad (12)$$

Here, by b_i, the coefficients in the expansion are indicated $f(p) = \sum_{i=0}^{\infty} b_i p^i$, and the sequence a_i has at least a factorial decrease. From the last equation it follows that, in the vicinity the point $q = c$, the Laplace-Borel transform of the equation's (12) right side is a holomorphic function. Let's apply the Laplace-Borel transform to the equation (12); it can be rewritten as

$$\begin{aligned} & q^k(q-c)\tilde{u}(q) + a_2^2\int_1^q\int_1^{q_2} q_1^{k-2}\tilde{u}(q_1)dq_1dq_2 + \ldots + \\ & + a_k^2\int_1^q \ldots \int_1^{q_2} \tilde{u}(q_1)dq_1\ldots dq_k + \\ & + B_{n-k}\sum_{j=0}^{k-i} b_1^j p^{\beta_1+(i+j-1)(n-k-1)}\left(\frac{1}{n-k-1}p^{n-k}\frac{d}{dp}\right)^{k+1-i-j}\tilde{u}(q) + \\ & + b_2^1 B_{n-k}p^{k(n-k-1)}\int_1^p \ldots \int_1^{p_2} p_1^{m_0}\hat{u}(p_1)dp_1\ldots dp_{\beta_2} = \\ & = a_0 q^{k-1}\ln q + \sum_{i=1}^{\infty} a_i q^{k-1+\frac{i}{n-k-1}} \end{aligned}$$

Here, as before, by $a_i^2, i = 2,\ldots,k$, some constants are denoted.

LEMMA. The asymptotics of the function $\tilde{u}(q)$ with $q \to c$ look like this

$$\sum_{j=0}^{n-k-1}(q-c)^{\frac{j}{n-k-1}}\sum_{i=0}^{\infty} A_i^j(q-c)^i + \sum_{i=0}^{\infty} C_i(q-c)^i\ln(q-c)$$

Proof.

Let's convert the last equation as follows

$$\begin{aligned} \tilde{u}(q) = & -\frac{a_2^2}{q^k(q-c)}\int_1^q\int_1^{q_2} q_1^{k-2}\tilde{u}(q_1)dq_1dq_2 - \ldots \\ & - \frac{a_{k-1}^2}{q^k(q-c)}\int_1^q \ldots \int_1^{q_2} q_1\tilde{u}(q_1)dq_1\ldots dq_{k-1} - \\ & - \frac{a_k^2}{q^k(q-c)}\int_1^q \ldots \int_1^{q_2} \tilde{u}(q_1)dq_1\ldots dq_k - \\ & - b_2^1\frac{1}{q^k(q-c)}B_{n-k} \times \\ & \times \sum_{j=0}^{k-i} b_1^j p^{\beta_1+(i+j-1)(n-k-1)}\left(\frac{1}{n-k}p^{n-k}\frac{d}{dp}\right)^{k+1-i-j}\tilde{u}(q) - \\ & - \frac{1}{q^k(q-c)}B_{n-k}p^{k(n-k-1)}b_2\int_1^p \ldots \int_1^{p_2} p_1^{m_0}\hat{u}(p_1)dp_1\ldots dp_{\beta_2} + \\ & + \frac{\ln q + \sum_{i=1}^{\infty} a_i q^{-1+\frac{i}{n-k-1}}}{q(q-c)} \end{aligned} \qquad (13)$$

To construct the asymptotics of the function $\tilde{u}(q)$, let's apply the method of successive approximations. Let's imagine that near the point $q = c$ there is the equation's (13) right side,

$$\frac{\ln q + \sum_{i=1}^{\infty} a_i q^{-1+\frac{i}{n-k-1}}}{q(q-c)} = \frac{C_1}{q-c} + A_1(q), \qquad (14)$$

here and after, by $A_i(q)$, we denote holomorphic function near the point $q = c$, ; by C_i, we denote the corresponding constants.

It is obvious that all the members in (13), containing multiple integrals, when substituting the free members into them, give the minor asymptotic members, for example, by substituting (14) into the first integral, we get

$$\begin{aligned} \int_0^q\int_0^q\left(\frac{C_1}{q_1-c} + A_1(q_1)\right)dq_1dq &= \int_0^q C_1\ln(q-c) + A_2(q)dq = \\ &= C_1(q-c)\ln(q-c) + A_3(q) \end{aligned}$$

At the next step, the method of successive approximations will be asymptotically look like $\frac{C_1}{2\times 3}(q-c)^3\ln(q-c)$; by continuing to apply the method of successive approximations, we obtain a series

$$\ln(q-c)\sum_{i=1}^{\infty} a_i(q-c)^{i-1}$$

Here, the sequence $a_i = \frac{C_1}{i!}$ is factorially decreasing. Let's show that the other members give minor asymptotic members. Obviously, if the multiplicity of the integrals is greater than or equal to two, we obtain the minor asymptotic members. Let's consider the last integral. Let's substitute the expression (14) into the last integral from the right side (13). If $\beta_2 \geq 2$, it is obvious that the function is a mollifier. Let $\beta_2 = 1$, then

$$\int p^{m_0} B_{n-k}^{-1} \frac{1}{q-c} dp =$$

$$= -\frac{1}{c(n-k-1)} \left(e^{\frac{c}{p^{n-k-1}}} p^{n-k+m_0} - (n-k+m_0) \int p^{n-k+m_0-1} e^{\frac{c}{p^{n-k-1}}} dp \right) =$$

$$= e^{\frac{c}{p^{n-k-1}}} \sum_{i=1}^{\infty} \frac{\left(1+\frac{m_0+1}{n-k-1}\right) \ldots \left(i-1+\frac{m_0+1}{n-k-1}\right)}{c(n-k-1)} p^{i(n-k-1)+m_0+1}.$$

Finally, we get that

$$B_{n-k} p^{k(n-k-1)} \int p^{m_0} B_{n-k}^{-1} \frac{1}{q-c} dp =$$

$$= \sum_{i=1}^{\infty} \frac{\left(1+\frac{m_0+1}{n-k-1}\right) \ldots \left(i-1+\frac{m_0+1}{n-k-1}\right)}{c(n-k-1) \Gamma\left(i+k+\frac{m_0+1}{n-k-1}\right)} (q-c)^{i+k+\frac{m_0+1}{n-k-1}-1} =$$

$$= (q-c)^{\frac{m_0+1}{n-k-1}} \times$$

$$\times \sum_{i=1}^{\infty} \frac{(q-c)^{i+k-1}}{c(n-k-1) \Gamma\left(\frac{m_0+1}{n-k-1}\right)\left(i+\frac{m_0+1}{n-k-1}\right) \ldots \left(i+k+\frac{m_0+1}{n-k-1}\right)}$$

It follows that, when applying the method of successive approximations, the last integral function of (13) gives a convergent series in powers $(q-c)$.

It remains to consider the member

$$B_{n-k} \sum_{j=0}^{k-i} b_1^j p^{\beta_1 + (i+j-1)(n-k-1)} \left(\frac{1}{n-k-1} p^{n-k} \frac{d}{dp} \right)^{k+1-i-j} \hat{u}(p) \quad (15)$$

Let's assume that $i > 1$. Since the equation is fulfilled,

$$B_{n-k} p^{\beta_1} p^{(i+j-1)(n-k-1)} \left(\frac{1}{n-k-1} p^{n-k} \frac{d}{dp} \right)^{k+1-i-j} \hat{u}(p) =$$

$$= B_{n-k} p^{(i+j-1)(n-k-1)+\beta_1} B_{n-k}^{-1} B_{n-k} \left(\frac{1}{n-k-1} p^{n-k} \frac{d}{dp} \right)^{k+1-i-j} \hat{u}(p) =$$

$$= B_{n-k} p^{(i+j-1)(n-k-1)+\beta_1} B_{n-k}^{-1} q^{k+1-i-j} \tilde{u}(q),$$

by substituting the right side (14) instead of $\tilde{u}(q)$, we get

$$B_{n-k} p^{(i+j-1)(n-k-1)+\beta_1} B_{n-k}^{-1} q^{k+1-i-j} \tilde{u}(q) =$$

$$= B_{n-k} p^{(i+j-1)(n-k-1)+\beta_1} B_{n-k}^{-1} \left(\frac{C_1}{q-c} + A_1(q) \right) =$$

$$= B_{n-k} p^{(i+j-1)(n-k-1)+\beta_1} e^{\frac{c}{p^{n-k-1}}} =$$

$$= \frac{(q-c)^{\frac{\beta_1}{n-k-1}+i+j-2}}{\Gamma\left(i+j-1+\frac{\beta_1}{n-k-1}\right)}$$

Since $\frac{\beta_1}{n-k-1} + i + j - 1 > 0$, the function (15) is a mollifier and, when applying the method of successive approximations, we get a convergent series in powers $(q-c)$. Let $i = 1, j = 0, \beta_1 < n-k-1$. This case must be considered separately. We have the member $B_{n-k} b_1^0 p^{\beta_1} \left(\frac{1}{n-k-1} p^{n-k} \frac{d}{dp} \right)^k \hat{u}(p).$

By replacement let's move the root $q = c$ to zero; in this case, zero will be a simple root. We will solve the resulting equation in the same way as it was done in [11] for equations with simple roots. As shown in this paper, the equation will be performed.

$$Br^{\beta_1} B^{-1} p^k \hat{u}(p) = \int_{\infty}^{p} (p-p')^{\frac{\beta_1}{n-k-1}-1} p'^k dp'$$

Since $\frac{\beta_1}{n-k-1} > 0$ and $k \geq 0$, the last integral will be a holomorphic function.

Finally, we get that the asymptotic member, corresponding to the root $q = c$, looks like this

$$\sum_{j=0}^{n-k-1} (q-c)^{\frac{j}{n-k-1}} \sum_{i=0}^{\infty} A_i^j (q-c)^i + \sum_{i=0}^{\infty} C_i (q-c)^i \ln(q-c)$$

Where A_i^j, C_i are corresponding constants. The lemma is proved.

Now let's consider the singularity at zero. We have the equation

$$q^{k+1}\tilde{u}(q) - cq^k \tilde{u}(q) + a_1 \int q^{k-1} \tilde{u}(q) dq +$$
$$+ a_2 \int_1^q \int_1^{q_2} q_1^{k-2} \tilde{u}(q_1) dq_1 dq_2 + \ldots$$
$$+ a_{k-1} \int_1^q \ldots \int_1^{q_2} q_1 \tilde{u}(q_1) dq_1 \ldots dq_{k-1} +$$
$$+ a_k \int_1^q \ldots \int_1^{q_2} \tilde{u}(q_1) dq_1 \ldots dq_k + \quad (16)$$
$$+ b_1 B_{n-k} p^{\beta_1} p^{i(n-k-1)} \left(p^{n-k} \frac{d}{dp} \right)^{k-i} \tilde{u}(q) +$$
$$+ B_{n-k} p^{k(n-k-1)} b_2 \int_1^p \ldots \int_1^{p_2} p_1^{m_0} \hat{u}(p_1) dp_1 \ldots dp_{\beta_2} =$$
$$= a_0 q^{k-1} \ln q + \sum_{i=1}^{\infty} a_i q^{k-1+\frac{i}{n-k-1}}$$

First, let's consider a special case, when the minor members are absent, i. e., when the equation looks like this

$$q^{k+1}\tilde{u}(q) - cq^k \tilde{u}(q) + a_1 \int q^{k-1} u(q) dq + a_2 \int_1^q \int_1^{q_2} q_1^{k-2} \tilde{u}(q_1) dq_1 dq_2 +$$
$$\ldots + a_k \int_1^q \ldots \int_1^{q_2} \tilde{u}(q_1) dq_1 \ldots dq_k = a_0 q^{k-1} \ln q + \sum_{i=1}^{\infty} a_i q^{k-1+\frac{i}{n-k-1}}.$$

The last equation, by differencing by q k times, is transformed to an equation with a conic singularity, the right side of which has a conormal asymptotics. As is known, the solution of such equations has conormal asymptotics. It would be natural to assume that, in the general case, i. e., with minor members, the solution will also look like a conormal asymptotics. Let's prove it.

Let's consider a group of members.

$$a_0 q^k \tilde{u}(q) + a_1 \int q^{k-1}\tilde{u}(q)dq + a_2 \int_1^q \int_1^{q_2} q_1^{k-2}\tilde{u}(q_1)dq_1 dq_2 + ... + a_{k-1}\int_1^q ... \int_1^{q_2} q_1 \tilde{u}(q_1)dq_1 ... dq_{k-1} + + a_k \int_1^q ... \int_1^{q_2} \tilde{u}(q_1)dq_1 ... dq_k \quad (17)$$

Let's show that this sum of integrals (17) can be represented as a single integral. Let's consider the sum

$$I_2\tilde{u}(q) = a_0 q^k \tilde{u}(q) + a_1 \int q^{k-1}\tilde{u}(q)dq$$

Let's denote $\tilde{u}(q) = q^\sigma \tilde{u}_1(q)$.

$$I_2\tilde{u} = a_0 q^k \tilde{u}(q) + a_1 \int q^{k-1}\tilde{u}(q)dq =$$

$$= a_0 q^{k+\sigma}\tilde{u}_1(q) + a_1 \int q^{k+\sigma-1}\tilde{u}_1(q)dq =$$

$$= \left(a_0 + \frac{a_1}{k+\sigma}\right) q^{k+\sigma}\tilde{u}_1(q) - \frac{a_1}{k+\sigma}\int q^{k+\sigma}\frac{d}{dq}\tilde{u}_1(q)dq$$

Let's choose σ so that $\frac{a_1}{k+\sigma} = -a_0$; we get

$$I_2\tilde{u} = a_0 \int q^{k+\sigma}\frac{d}{dq}\tilde{u}_1(q)dq = a_0 \int q^{k+\sigma}\frac{d}{dq}\frac{\tilde{u}(q)}{q^\sigma}dq, \quad (18)$$

We have shown that, in the case, when the sum (17) consists of two members, it can be represented as a single integral (18).

Now let's consider a group of three members.

$$I_3\tilde{u} = a_0 q^k \tilde{u}(q) + $$
$$+ a_1 \int q^{k-1}\tilde{u}(q)dq + a_2 \int_1^q \int_1^q q_1^{k-2}\tilde{u}(q_1)dq_1 dq =$$
$$= \left(a_0 + \frac{a_1}{k+\sigma}\right) q^{k+\sigma}\tilde{u}_1(q) +$$
$$+ \int\left(-\frac{a_1}{k+\sigma}q^{k+\sigma}\frac{d}{dq}\tilde{u}_1(q) + a_2 \int_1^q q_1^{k-2+\sigma}\tilde{u}_1(q_1)dq_1\right)dq.$$

Let's introduce the notation $\tilde{u}_2(q) = \frac{d}{dq}\tilde{u}_1(q)$. Since

$$-\frac{a_1}{k+\sigma}q^{k+\sigma}\frac{d}{dq}\tilde{u}_1(q) + a_2 \int q^{k-2+\sigma}\tilde{u}_1(q)dq =$$
$$== -\frac{a_1}{k+\sigma}q^{k+\sigma}\tilde{u}_2(q) + \frac{a_2}{k-1+\sigma}q^{k-1+\sigma}\tilde{u}_1(q) -$$
$$-\frac{a_2}{k-1+\sigma}\int q^{k-1+\sigma}\tilde{u}_2(q)dq,$$

then

$$I_3\tilde{u}(q) =$$
$$= \left(a_0 + \frac{a_1}{k+\sigma}\right) q^{k+\sigma}\tilde{u}_1(q) + \int \frac{a_2}{k-1+\sigma}q^{k-1+\sigma}\tilde{u}_1(q)dq +$$
$$+ \int_1^q (-\frac{a_1}{k+\sigma}q^{k+\sigma}\tilde{u}_2(q) -$$
$$-\frac{a_2}{k-1+\sigma}\int_1^q q^{k-1+\sigma}\tilde{u}_2(q_1)dq_1)dq.$$

Since the equation is fulfilled,

$$\left(a_0 + \frac{a_1}{k+\sigma}\right) q^{k+\sigma}\tilde{u}_1(p) + \frac{a_2}{k-1+\sigma}\int q^{k-1+\sigma}\tilde{u}_1(q)dq =$$
$$= \left(a_0 + \frac{a_1}{k+\sigma} + \frac{a_2}{k-1+\sigma}\frac{1}{k+\sigma}\right) q^{k+\sigma}\tilde{u}_1 -$$
$$-\frac{a_2}{k-1+\sigma}\frac{1}{k+\sigma}\int q^{k+\sigma}\frac{d}{dq}\tilde{u}_1(q)dq,$$

and, since we can choose σ so that equality is fulfilled,

$$\frac{a_1}{k+\sigma} + \frac{a_2}{k-1+\sigma}\frac{1}{k+\sigma} = -a_0,$$

by denoting $\tilde{u}_2(q) = q^{\sigma_1}\tilde{u}_3(q)$ we get the equation

$$I_3\tilde{u} =$$
$$= \left(-\frac{a_1}{k+\sigma} - \frac{a_2}{k-1+\sigma}\frac{1}{k+\sigma}\right)\int q^{k+\sigma+\sigma_1}\tilde{u}_3(q)dq -$$
$$-\frac{a_2}{k-1+\sigma}\frac{1}{k+\sigma+\sigma_1}\int_1^q (\int_1^{q_2} q_1^{k+\sigma+\sigma_1-1}\tilde{u}_3(q_1)dq_1)dq_2 =$$
$$= -\int(\left(\frac{a_1}{k+\sigma} + \frac{a_2}{k-1+\sigma}\frac{1}{k+\sigma} + \frac{a_2}{k-1+\sigma}\frac{1}{k+\sigma+\sigma_1}\right) q^{k+\sigma+\sigma_1}\tilde{u}_3(q)dq -$$
$$-\frac{a_2}{k-1+\sigma}\frac{1}{k+\sigma+\sigma_1}\int q^{k+\sigma+\sigma_1}\frac{d}{dq}\tilde{u}_3(q)dq$$

let's choose σ_1 so that equation is fulfilled

$$\frac{a_1}{k+\sigma} + \frac{a_2}{k-1+\sigma}\frac{1}{k+\sigma} + \frac{a_2}{k-1+\sigma}\frac{1}{k+\sigma+\sigma_1} = 0.$$

We finally get

$$I_3\tilde{u}(q)=a_0q^k\tilde{u}(q)+a_1\int q^{k-1}\tilde{u}(q)dq+a_2\int_1^q\int_1^{q_2} q_1^{k-2}\tilde{u}(q_1)dq_1dq_2=$$

$$=a_0\int_1^q\int_1^{q_2} q_1^{k+\sigma+\sigma_1}\frac{d}{dq_1}\tilde{u}_3(q_1)dq_1dq_2=$$

$$=a_0\int_1^q\int_1^{q_2} q_1^{k+\sigma+\sigma_1}\frac{d}{dq_1}\frac{\tilde{u}_2(q_1)}{q_1^{\sigma_1}}dq_1dq_2=$$

$$=a_0\int_1^q\int_1^{q_2} q_1^{k+\sigma+\sigma_1}\frac{d}{dq_1}\frac{1}{q_1^{\sigma_1}}\frac{d}{dq_1}\frac{1}{q_1^{\sigma}}\tilde{u}(q_1)dq_1dq_2$$

Like the previous case, we get

$$I_k\tilde{u}=a_0q^k\tilde{u}(q)+a_1\int q^{k-1}\tilde{u}(q)dq+$$

$$+a_2\int_1^q\int_1^{q_2} q_1^{k-2}\tilde{u}(q_1)dq_1dq_2+\ldots+a_i\int_1^q\ldots\int_1^{q_2}\tilde{u}(q_1)dq_1\ldots dq_k= \tag{19}$$

$$=a_0\int_1^q\ldots\int_1^{q_2} q_1^{k+\sigma_1+\sigma_2+\ldots+\sigma_k}\frac{d}{dq_1}\frac{1}{q_1^{\sigma_k}}\ldots$$

$$\frac{d}{dq_1}\frac{1}{q_1^{\sigma_2}}\frac{d}{dq_1}\frac{1}{q_1^{\sigma_1}}\tilde{u}(q_1)dq_1\ldots dq_{k1}$$

Let’s substitute the obtained integral in the equation (16); it looks like this

$$q^{k+1}q^{\sigma_1}\int_1^q\ldots\int_1^{q_4} q_3^{\sigma_{k-1}}\int_1^{q_3} q_2^{\sigma_k}\int_1^{q_2}\tilde{u}_k(q_1)dq_1dq_2\ldots dq_k-$$

$$-c\int_1^q\ldots\int_1^{q_2} q_1^{k+\sum_{i=1}^{k}\sigma_i}\tilde{u}_k(q_1)dq_1\ldots dq_k+$$

$$+b_1B_{n-k}p^{\beta_1}p^{i(n-k-1)}\left(p^{n-k}\frac{d}{dp}\right)^{k-i}B_{n-k}^{-1}\times$$

$$\times\left(q^{\sigma_1}\int_1^q\ldots\int_1^{q_4} q_3^{\sigma_{k-1}}\int_1^{q_3} q_2^{\sigma_k}\int_1^{q_2}\tilde{u}_k(q_1)dq_1\ldots dq_k\right)+ \tag{20}$$

$$+B_{n-k}p^{k(n-k-1)}b_2\int_1^p\ldots\int_1^{p_2} p_1^{m_0}B_{n-k}^{-1}\times$$

$$\times\left(q^{\sigma_1}\int_1^q\ldots\int_1^{q_4} q_3^{\sigma_{k-1}}\int_1^{q_3} q_2^{\sigma_k}\int_1^{q_2}\tilde{u}_k(q_1)dq_1\ldots dq_k\right)dp_1\ldots dp_{\beta_2}=$$

$$=a_0q^{k-1}\ln q+\sum_{i=1}^{\infty}a_iq^{k-1+\frac{i}{n-k-1}}$$

Let's transform the equation (20), expressing $\tilde{u}_k(q)$

$$\tilde{u}_k(q)=-\frac{1}{c}q^{-k-\sum_{i=1}^{k}\sigma_i}\left(\frac{d}{dq}\right)^k\left(a_0q^{k-1}\ln q+\sum_{i=1}^{\infty}a_iq^{k-1+\frac{i}{n-k-1}}\right)+$$

$$+\frac{1}{c}q^{-k-\sum_{i=1}^{k}\sigma_i}\left(\frac{d}{dq}\right)^k\int_1^q\ldots\int_1^{q_4} q_3^{\sigma_{k-1}}\int_1^{q_3} q_2^{\sigma_k}\int_1^{q_2}\tilde{u}_k(q_1)dq_1dq_2\ldots dq_k+$$

$$+\frac{b_1}{c}q^{-k-\sum_{i=1}^{k}\sigma_i}\left(\frac{d}{dq}\right)^k B_{n-k}p^{\beta_i}p^{i(n-k-1)}\left(p^{n-k}\frac{d}{dp}\right)^{k-i}\times$$

$$\times B_{n-k}^{-1}\left(q^{\sigma_1}\int_1^q\ldots\int_1^{q_4} q_3^{\sigma_{k-1}}\int_1^{q_3} q_2^{\sigma_k}\int_1^{q_2}\tilde{u}_k(q_1)dq_1\ldots dq_k\right)+ \tag{21}$$

$$+\frac{b_2}{c}q^{-k-\sum_{i=1}^{k}\sigma_i}\left(\frac{d}{dq}\right)^k B_{n-k}p^{k(n-k-1)}\times$$

$$\times\int_1^p\ldots\int_1^{p_2} p_1^{m_0}B_{n-k}^{-1}\times$$

$$\times\left(q^{\sigma_1}\int_1^q\ldots\int_1^{q_4} q_3^{\sigma_{k-1}}\int_1^{q_3} q_2^{\sigma_k}\int_1^{q_2}\tilde{u}_k(q_1)dq_1\ldots dq_k\right)dp_1\ldots dp_{\beta_2}$$

To find the asymptotics of the function $\tilde{u}_k(q)$ at $q\to 0$, let's apply the method of successive approximations.

LEMMA. The asymptotics of the function $\tilde{u}_k(q)$ *at* $q\to 0$ *is conormal*

Proof.

The free member in (21), at $q\to 0$, has asymptotics

$$q^{-k-\sum_{i=1}^{k}\sigma_i}\left(\frac{d}{dq}\right)^k\left(a_0q^{k-1}\ln q+\sum_{i=1}^{\infty}a_iq^{k-1+\frac{i}{n-k-1}}\right)=\sum_{i=0}^{\infty}a_i^1q^{-1+\frac{i}{n-k-1}-k-\sum_{j=1}^{k}\sigma_j}$$

Let's introduce the notion $\alpha=-1+\frac{1}{n-k-1}-k-\sum_{i=1}^{k-m}\sigma_i$; we will assume that $\alpha+\sum_{i=1}^{k-m}\sigma_i+m\neq -1$, with all $1\le m\le k$; by substituting the free member into the first integral of (21), we get.

$$I_1q^{\alpha}=q^{-k-\sum_{i=1}^{k}\sigma_i}\left(\frac{d}{dq}\right)^k q^{k+1}q^{\sigma_1}\int_1^q\ldots\int_1^{q_4} q_3^{\sigma_{k-1}}\int_1^{q_3} q_2^{\sigma_k}\int_1^{q_2} q_1^{\alpha}dq_1dq_2\ldots dq_k=$$

$$=\frac{1}{(\alpha+1)(\alpha+\sigma_k+2)\ldots\left(\alpha+\sum_{i=1}^{k}\sigma_i+k\right)}q^{-k-\sum_{i=1}^{k}\sigma_i}\left(\frac{d}{dq}\right)^k q^{1+\alpha+2k+\sum_{i=1}^{k}\sigma_i}=$$

$$=q^{1+\alpha}\frac{\left(1+\alpha+2k+\sum_{i=1}^{k}\sigma_i\right)\left(\alpha+2k+\sum_{i=1}^{k}\sigma_i\right)\ldots\left(\alpha+\sum_{i=1}^{k}\sigma_i+k\right)}{(\alpha+1)(\alpha+\sigma_k+2)\ldots\left(\alpha+\sum_{i=1}^{k}\sigma_i+k\right)}.$$

From the last equation, it follows that, when applying the method of successive approximations, the function I_1 will correspond to a convergent series in powers q. If, for some $1\le m\le k$, the equation $\alpha+\sum_{i=1}^{k-m}\sigma_i+m=-1$ is satisfied, the asymptotics, corresponding to the function I_1, will look like this $q^{\alpha+1-k-\sum_{i=1}^{k}\sigma_i}\sum_{i=0}^{k}a_i(\ln q)^i$; in other words, the asymptotics will be conormal.

Let's consider the second member from (21) and substitute the free member in it

$$I_2 q^\alpha =$$
$$= q^{-k-\sum_{i=1}^{k}\sigma_i}\left(\frac{d}{dq}\right)^k B_{n-k}p^{\beta_0}p^{i(n-k-1)}\left(p^{n-k}\frac{d}{dp}\right)^{k-i}\times$$
$$\times B_{n-k}^{-1}\int_1^q\ldots\int_1^{q_4}q_3^{\sigma_{k-1}}\int_1^{q_3}q_2^{\sigma_k}\int_1^{q_2}q_1^{\alpha}dq_1dq_2\ldots dq_k =$$
$$= C_2 q^{-k-\sum_{i=1}^{k}\sigma_i}\left(\frac{d}{dq}\right)^k B_{n-k}p^{\beta_0}B_{n-k}^{-1}q^{\alpha+2k+\sum_{i=1}^{k}\sigma_i}$$

here

$$C_2 = \frac{1}{(\alpha+1)(\alpha+\sigma_k+2)\ldots\left(\alpha+\sum_{i=1}^{k}\sigma_i+k\right)}\times$$
$$\times\frac{1}{\left(\alpha+\sum_{i=1}^{k}\sigma_i+2k-i+1\right)\ldots\left(\alpha+\sum_{i=1}^{k}\sigma_i+2k+1\right)}.$$

Since

$$B_{n-k}^{-1}q^{\alpha+2k-\sum_{i=1}^{k}\sigma_i} =$$
$$= \frac{(n-k-1)\Gamma\left(\alpha+2k-\sum_{i=1}^{k}\sigma_i+1\right)\sin\pi\left(\alpha+2k-\sum_{i=1}^{k}\sigma_i+1\right)}{\pi}\times$$
$$\times p^{\left(\alpha+2k-\sum_{i=1}^{k}\sigma_i+1\right)(n-k-1)},$$

then by introducing the notion

$$C_3 =$$
$$= C_2\frac{(n-k-1)\Gamma\left(\alpha+2k+\sum_{i=1}^{k}\sigma_i+1\right)\sin\pi\left(\alpha+2k+\sum_{i=1}^{k}\sigma_i+1\right)}{\pi}$$

we get

$$I_2 q^\alpha = C_3 q^{-k-\sum_{i=1}^{k}\sigma_i}\left(\frac{d}{dq}\right)^k B_{n-k}p^{\left(\alpha+2k+\sum_{i=1}^{k}\sigma_i+1\right)(n-k-1)+\beta_0}.$$

For definiteness, we assume that $\alpha+2k+\sum_{i=1}^{k}\sigma_i+1+\frac{\beta_0}{n-k-1}\notin Z$; we obtain the equation

$$I_2 q^\alpha = C_4 q^{\alpha+\frac{\beta_0}{n-k-1}}, \qquad (22)$$

where

$$C_4 =$$
$$= \frac{\sin\pi\left(\alpha+2k+\sum_{i=1}^{k}\sigma_i+1\right)}{\sin\pi\left(\alpha+2k+\sum_{i=1}^{k}\sigma_i+1+\frac{\beta_0}{n-k-1}\right)}\frac{\Gamma\left(\alpha+2k+\sum_{i=1}^{k}\sigma_i+1\right)}{\Gamma\left(\alpha+2k+\sum_{i=1}^{k}\sigma_i+1+\frac{\beta_0}{n-k-1}\right)}\times$$
$$\times\frac{\left(\alpha+2k+\sum_{i=1}^{k}\sigma_i+1+\frac{\beta_0}{n-k-1}\right)\ldots\left(\alpha+k+\sum_{i=1}^{k}\sigma_i+1+\frac{\beta_0}{n-k-1}\right)}{(\alpha+1)(\alpha+\sigma_k+2)\ldots\left(\alpha+\sum_{i=1}^{k}\sigma_i+k\right)}\times$$
$$\times\frac{1}{\left(\alpha+\sum_{i=1}^{k}\sigma_i+2k-i+1\right)\ldots\left(\alpha+\sum_{i=1}^{k}\sigma_i+2k+1\right)}$$

It follows that, when applying the method of successive approximations to the equation (21), the function I_2 also induces a convergent series in powers q. A similar result can be obtained for the last integral in (21).

We obtain the asymptotic behavior of the function $\tilde{u}(q)$ with $q\to 0$; it is a conormal asymptotics. The lemma is proved. As is known [5], the inverse Laplace-Borel transform of conormal asymptotics is also conormal asymptotics. It remains to find the inverse transform of the asymptotic member, corresponding to the singular point $q=c$.

$$B_{n-k}^{-1}\left(\sum_{j=0}^{n-k-1}(q-c)^{\frac{j}{n-k-1}}\sum_{i=0}^{\infty}A_i^j(q-c)^i+\sum_{i=0}^{\infty}C_i(q-c)^i\ln(q-c)\right)=$$
$$= e^{\frac{c}{p^{n-k-1}}}p^\sigma\sum_{i=0}^{\infty}C_i p^i$$

We will find the asymptotic member, corresponding to the root $q=c$. In [6], the asymptotics of the function's inverse Laplace-Borel transform $p^\sigma e^{\sum_{i=0}^{n-2}\frac{\beta_i}{p^{n-k-i-1}}}$ with $r\to 0$ was found; in this work, it has been shown that

$$B_1^{-1}p^\sigma e^{\sum_{i=0}^{n-2}\frac{\beta_i}{p^{n-i-1}}}\approx\sum_{j=1}^{n}e^{\sum_{i=1}^{n-1}\frac{\alpha_i^j}{r^{\frac{i}{n}}}}r^{\frac{\sigma_j}{n}}\sum_{k=0}^{\infty}c_k^j r^{\frac{k}{n}}$$

where α_{n-1}^j are polynomial roots $p^n+\left(\frac{n}{n-1}\right)^n(1-n)\beta_0$, and $\sigma_i,\sigma,$ and $\alpha_i^j,\ i=1,\ldots,n-2$ are corresponding numbers. From this formula, it follows that the asymptotic solution of the equation looks like this

$$u(r)=\sum_{j=1}^{n-k}e^{\sum_{i=1}^{n-k-1}\frac{\alpha_i^j}{r^{\frac{i}{n-k}}}}r^{\frac{\sigma_i}{n-k}}\sum_{l}^{\infty}A_l^j r^{\frac{l}{n-k}}+\sum_{j=0}^{k_0}(\ln r)^j r^{\alpha_j}\sum_{i=0}^{\infty}b_i^j r^i,$$

The theorem is proved.

Previously, we assumed that the main symbol of the differential function has one root, even if it is not. Let the main symbol have two roots, i. e.

$$H_0(p)=p^n+c_1p^{n-1}+c_2p^{n-2}+\ldots+c_np^{n-n_1}=p^{n-n_1}(p-b)^{n_1},$$

Here, c_i are corresponding numbers. In this case, instead of the equation (7), we will have the equation

$$\left(-r^2\frac{d}{dr}\right)^n u + c_1\left(-r^2\frac{d}{dr}\right)^{n-1} u + c_2\left(-r^2\frac{d}{dr}\right)^{n-2} u + ...$$
$$+ c_n\left(-r^2\frac{d}{dr}\right)^{n-n_1} u + a_0 r\left(-r^2\frac{d}{dr}\right)^{k} u + a_1 r^2\left(-r^2\frac{d}{dr}\right)^{k-1} u +$$
$$+ a_2 r^3\left(-r^2\frac{d}{dr}\right)^{k-2} u + ... + a_{i-1} r^i\left(-r^2\frac{d}{dr}\right)^{k-i+1} + ... + a_k r^{k+1} u +$$
$$b_1 r^i\left(-r^2\frac{d}{dr}\right)^{k-i+1+\beta_1} u + b_2 r^{k+1+\beta_2}\left(-r^2\frac{d}{dr}\right)^{m_0} u = 0$$

Let's make the Laplace-Borel transform

$$\begin{gathered} p^{n-n_1}(p+b)^{n_1}\hat{u}(p) + a_0\int p^k\hat{u}(p)dp + \\ + a_1\int_1^p\int_1^{p_2} p_1^{\,k-1}\hat{u}(p_1)dp_1dp_2 + ... \\ + a_{i-1}\int_1^p ... \int_1^{p_2} p_1^{\,k-i+1}\hat{u}(p_1)dp_1...dp_i + ... \\ + a_k\int_1^p ... \int_1^{p_2}\hat{u}(p_1)dp_1...dp_{k+1} + \\ + b_1\int_1^p ... \int_1^{p_2} p_1^{\,k-i+1+\beta_1}\hat{u}(p_1)dp_1...dp_i + \\ + b_2\int_1^p ... \int_1^{p_2} p_1^{\,k-i+1+\beta_1}\hat{u}(p_1)dp_1...dp_{k+\beta_2+1} = f(p) \end{gathered} \quad (23)$$

Let's find the asymptotic solution of the equation (23) with $p \to 0$; for this, let's rewrite the equation (23) like this

$$\begin{gathered} p^{n-n_1}\hat{u}(p) + \frac{a_0}{(p+b)^{n_1}}\int p^k\hat{u}(p)dp + \\ + \frac{a_1}{(p+b)^{n_1}}\int_1^p\int_1^{p_2} p_1^{\,k-1}\hat{u}(p_1)dp_1dp_2 + ... \\ + \frac{a_{i-1}}{(p+b)^{n_1}}\int_1^p ... \int_1^{p_2} p_1^{\,k-i+1}\hat{u}(p_1)dp_1...dp_i + ... \\ + \frac{a_k}{(p+b)^{n_1}}\int_1^p ... \int_1^{p_2}\hat{u}(p_1)dp_1...dp_{k+1} + \\ + \frac{b_1}{(p+b)^{n_1}}\int_1^p ... \int_1^{p_2} p_1^{\,k-i+1+\beta_1}\hat{u}(p_1)dp_1...dp_i + \\ + \frac{b_2}{(p+b)^{n_1}}\int_1^p ... \int_1^{p_2} p_1^{\,k-i+1+\beta_1}\hat{u}(p_1)dp_1...dp_{k+\beta_2+1} = \frac{f(p)}{(p+b)^{n_1}} \end{gathered} \quad (24)$$

Since the functions $\frac{a_i}{(p+b)^{n_1}}$ have no singularities at zero, the asymptotics of the solution to the equation (22) near the point $p=0$ is obtained, using the method of successive approximations, similar to how it was done for the equation (9). To find the asymptotics, corresponding to the root $-b$, this root should be shifted to zero; this can be done by replacing $u(r) = e^{-\frac{b}{r}}u_1(r)$ and finding the asymptotics at zero as it is done above.

Let's note that, if $k \geq n - n_1$, the solution will not have singularity; in case $k = n - n_1 - 1$, the asymptotics of the solution is conormal.

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