Exact solution of the three-state generalized double-chain Potts model

P. V. Khrapov, G. A. Skvortsov

***Abstract* — An exact analytical solution of generalized three-state double-chain Potts model with multi-spin interactions which are invariant under cyclic shift of all spin values is obtained. The partition function in a finite cyclically closed strip of length L, as well as the free energy, internal energy, entropy and heat capacity in thermodynamic limit are calculated using transfer-matrix method. Partial magnetization and susceptibility are suggested as the generalization of usual physical characteristics of a system.**

Proposed model can be interpreted as a generalized version of standard Potts model (which has Hamiltonian expressed through Kronecker symbols) and clock model (with Hamiltonian expressed through cosines). Considering a particular example of the model with plenty of forces, model's ground states are found, figures of its thermodynamic characteristics and discussed their behaviour at low temperature are shown.

***Keywords* — Potts model, Ising model, lattice, Hamiltonian, transfer-matrix, exact solution, partition function, free energy, internal energy, heat capacity, magnetization, susceptibility.**

I. Introduction

Lattice spin models, which originally appeared to explain ferromagnetic phenomena, have now been successfully used to study collective behavior in general [1]. For instance, in [2] 21-state Potts model was used to study the protein folding process. Generalized Potts models with large number of states appear in many areas, including physics (site percolation in the lattice gas [3]), biology (polymer gelation [4], studying a cancerous tumor [5]), algebra, computer science, sociology, medicine and so on (see [6], [7]).

An important role in the development of the phase transitions theory was played by exact analytical solutions obtained for a few number of lattice spin models. Important technical improvement in the mathematical apparatus of lattice spin models was made in 1941 [8], when the Ising model was formulated on matrix language and therefore calculation of a partition function was reduced to the search of transfer-matrix's eigenvalues.

The Ising model is the Potts model with two states. In 1944 L. Onsager [9] obtained an analytical solution of the Ising model with nearest-neighbors (i.e. with two coupling constants: horizontal J_h and vertical J_v ones) on two-dimensional square lattice. Phase transition temperature of this model was first discovered by Kramers and Wannier [8]: $(e^K-1)(e^L-1)=2,$ where $K=\frac{J_h}{k_BT}, L=\frac{J_v}{k_BT},$ and Potts, generalizing the formula above, showed, that the Potts model has a phase transition at $(e^K-1)(e^L-1)=q,$ where q is the number of states [10].

There is a variety of Potts models: standard, generalized, clock, Ashkin-Teller, etc. Exact solutions were obtained for 3-state and 4-state single-chain Potts models with nearest-neighbors [11], for special cases of 6-state [13] and 12-state [12] models. In [14] an exact analytical solution for double-chain Potts model with 10 forces in a unit cell and arbitrary integer q was obtained. In [15] disorder solutions for generalized Potts model were calculated.

Potts model can be investigated using graph theory methods (see [16], [17]), usually with the help of Tutte polynomials. In 1972 Fortuin and Kasteleyn [18] showed the relationship between Potts model and Tutte polynomials and introduced the random cluster model which generalizes Potts model on arbitrary positive non-integer q. In [27], [29] Potts models with different boundary conditions were considered, in [28] some exact results on Potts model with external field were obtained. In [23] authors derived an exact solution of gauge Z_4 Potts model on square and triangular lattices. Cluster properties and bound states of the Yang-Mills model with compact Abelian gauge group were studied in [19].

Recently, papers have appeared in which the Ising and Potts models are investigated experimentally: using non-equilibrium quantum condensates [21], simulator of quantum computer [22].

Potts model can be investigated on one-, two-, and three-dimensional lattices: square, triangular, kagome, body-centered cubic, Bethe and honeycomb ones [31]. There are works in which authors calculate zeros of partition function ([24], [26]), correlation functions [20], investigate ground states [34] and so on.

Potts model is used in many fields, moreover, researchers often need non-standard Potts models with quite specific Hamiltonians (see [6], p. 53). That is why obtaining an exact solution for wide class of Hamiltonians is an important problem.

In this paper, three-state double-chain Potts model with cyclic boundary conditions is considered. An exact analytical solution of this model in a finite strip of length L is obtained

Manuscript received February 3, 2025.
P. V. Khrapov – Bauman Moscow State Technical University (5/1 2-nd Baumanskaya St., Moscow 105005, Russia),
ORCID: https://orcid.org/0000-0002-6269-0727 ,
e-mail: pvkhrapov@gmail.com , khrapov@bmstu.ru.
G. A. Skvortsov – Bauman Moscow State Technical University (5/1 2-nd Baumanskaya St., Moscow 105005, Russia),
ORCID: https://orcid.org/0009-0005-5140-785X,
e-mail: skvortsovga@student.bmstu.ru , skvortsov.grisha@list.ru

using transfer-matrix method and analytical expressions for its physical characteristics in thermodynamic limit are derived using Cardano formula (see section and [38]).

In section II the model is introduced, its Hamiltonian is formulated through Kronecker symbols, and main results are listed. Some generalization of Potts model's characteristics such as the partial magnetization and susceptibility are presented. These generalized characteristics allow to represent the mean value of an arbitrary random function of the spin variable $\langle\phi(\sigma)\rangle$, where σ is some spin of the lattice, as well as the covariances of such functions in the form of their linear combinations.

In section III the clock version of the original Hamiltonian is introduced.

In section IV an example of model given being analyzed in detail: plots of free energy, entropy, heat capacity, internal energy and partial susceptibility are shown, a table of ground states is provided.

Section V contains proof of theorem 1.

In Appendix the Cardano formula for solving an equation of the third degree is discussed.

II. Generalized Potts model with multi-spin interactions invariant under cyclic shift of all spin values

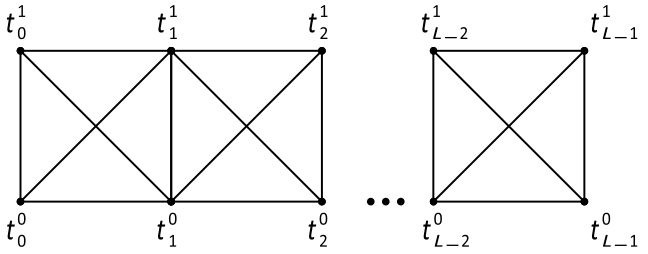


Fig. 1. Lattice $\mathsf{L}_{2,L}$, only two-site couplings of the model are displayed. Transfer-matrix Θ propagates from left to right.

Consider cyclically closed double-chain Potts model in a finite strip of legth L : on the lattice $\mathsf{L}_{2,L}$: $\mathsf{L}_{2,L} = \{t_i^j \mid i = 0,1,\ldots,L-1, t_L^j \equiv t_0^j, j = 0,1\}$. Each point t_i^j contains a particle. The state of the particle is determined by the value of the spin $\sigma_i^j \in X = \{0,1,\ldots,q-1\}, \sigma \in X^{\mathsf{L}_{2,L}}$ and the Hamiltonian of $\mathsf{L}_{2,L}$ has the form:

$$\mathsf{H}_L(\sigma) = \sum_{i=0}^{L-1} H\left(\sigma_i^0, \sigma_i^1; \sigma_{i+1}^0, \sigma_{i+1}^1\right), \qquad (1)$$

Elementary Hamiltonian H of a cell $\Omega_i = \{t_i^0, t_i^1, t_{i+1}^0, t_{i+1}^1\}$ takes the form:

$$H\left(\sigma|_{\Omega_i}\right) = H\left(\sigma_i^0, \sigma_i^1; \sigma_{i+1}^0, \sigma_{i+1}^1\right) =$$

$$-\left(\sum_{\{\mu_1,\mu_2,\mu_3\}\in X^3} J_{\mu_1,\mu_2,\mu_3}\delta_{\sigma_0,\mu_1+\sigma_1,\mu_2+\sigma_2,\mu_3+\sigma_3} + \sum_{\mu\in X} h_\mu \sum_{i=0}^{3}\delta_{\mu,\sigma_i}\right) \qquad (2)$$

where $\sigma_0 \equiv \sigma_i^0, \sigma_1 \equiv \sigma_i^1, \sigma_2 \equiv \sigma_{i+1}^0, \sigma_3 \equiv \sigma_{i+1}^1$,

$$\delta_{a_1,a_2,\ldots,a_s} = \begin{cases} 1, & \text{if } a_r \equiv a_l (mod\ q), r = 1,2,\ldots s, l = 1,\ldots s \\ 0, & \text{otherwise.} \end{cases}$$

$\delta_{a_1,a_2,\ldots,a_s}$ denotes the Kronecker symbol and the last term of Hamiltonian (2) corresponds to the external field $\vec{h} = \{h_0, h_1, h_2\}$.

Note that in further transformations, the Boltzmann constant k_B is set equal to unity, while temperature T and interactions' constants J_{μ_1,μ_2,μ_3} will be measured in the same units as is usually done in the theory of low-dimensional systems.

Let us introduce the transfer-matrix Θ with matrix elements

$$\langle\sigma_0, \sigma_1 | \Theta | \sigma_2, \sigma_3\rangle = exp\left[H(\sigma_0, \sigma_1; \sigma_2, \sigma_3)/T\right]$$

and such a structure (see Figure 2):

		0			1			2			σ_3
		0	1	2	0	1	2	0	1	2	σ_2
0	0										
	1										
	2										
1	0										
	1										
	2										
2	0										
	1										
	2										
σ_1	σ_0										

Fig. 2. The structure of the transfer-matrix Θ

Then the partition function $Z_L = \sum_{\sigma\in X^{G_L}} e^{-\frac{\mathsf{H}_L(\sigma)}{T}}$ can be represented as $Z_L = \mathrm{Tr}\left(\Theta^L\right)$.

Let $q = 3$ and the external field is equal to zero, i.e. spins σ_j^i take three values 0,1,2 and $\vec{h} = \vec{0}$. Then Hamiltonian (2) is invariant under cyclic shift of all spin values:

$$H(i+1, j+1; k+1, l+1) = H(i, j; k, l), \qquad (3)$$

wherein addition is performed in $\square_3$.

Corresponding transfer-matrix takes the form:

$$\Theta = \begin{pmatrix} a & d & e & f & g & h & i & j & k \\ l & b & m & n & o & p & q & r & s \\ t & u & c & v & w & x & y & z & A \\ A & y & z & c & t & u & x & v & w \\ k & i & j & e & a & d & h & f & g \\ s & q & r & m & l & b & p & n & o \\ o & p & n & r & s & q & b & m & l \\ w & x & v & z & A & y & u & c & t \\ g & h & f & j & k & i & d & e & a \end{pmatrix} \qquad (4)$$

Due to condition (3) transfer-matrix Θ (4) commutes with the permutation matrix Щ:

$$\text{Щ} = \begin{pmatrix} 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0 \end{pmatrix} \quad (5)$$

This circumstance allows us to find all its eigenvalues.

Theorem 1

Partition function of considered model in a cyclically closed finite strip of length L *can be presented as*:

$Z_L = \sum_{\varepsilon\in E}\sum_{k=1}^{3}\lambda_{\varepsilon,k}^{L}$, *where* $E = \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$ *is the set of cubic roots of* 1, $\lambda_{\varepsilon,k}$ *are eigenvalues of matrices*:

$$\Theta_\varepsilon = \begin{pmatrix} a+g\varepsilon+k\varepsilon^2 & d+h\varepsilon+\tilde{i}\varepsilon^2 & e+f\varepsilon+j\varepsilon^2\\ l+o\varepsilon+s\varepsilon^2 & b+p\varepsilon+q\varepsilon^2 & m+n\varepsilon+r\varepsilon^2\\ t+w\varepsilon+A\varepsilon^2 & u+x\varepsilon+y\varepsilon^2 & c+v\varepsilon+z\varepsilon^2 \end{pmatrix} \quad (6)$$

The principal (single largest real) eigenvalue of the matrix Θ_1 *determined from its secular equation, can be expressed in radicals* (see section VII) *and coincides with the prinicipal eigenvalue of the transfer-matrix* Θ. *Furthermore,* $\overline{\lambda}_{\varepsilon,k} = \lambda_{\overline{\varepsilon},k}, k = 1,2,3.$

Main model's thermodynamic characteristics in a strip of length L can be expressed through its partition function [39]:

- Free energy is equal to:

$$f(T) = -\frac{T}{N}\ln Z_L(T),$$

where $N = 2L$ - the number of lattice's sites.

- Internal energy is equal to:

$$u(T) = -T^2\frac{\partial}{\partial T}\left(\frac{f(T)}{T}\right).$$

- Entropy is expressed as:

$$S(T) = -\frac{\partial}{\partial T}f(T),$$

- Heat capacity is:

$$c(T) = \frac{\partial}{\partial T}u(T) = -2T\frac{\partial}{\partial T}\left(\frac{f(T)}{T}\right) - T^2\frac{\partial^2}{\partial T^2}\left(\frac{f(T)}{T}\right).$$

Definition 1

Let random variable M_μ be $\mathsf{M}_\mu = \sum_{i=0}^{L-1}\sum_{j=0}^{1}\delta_{\mu,\sigma_i^j}$.

Definition 2

Partial magnetization m_μ and partial susceptibility χ_{μ_0,μ_1} of double-chain Potts model are determined as:

$$m_\mu = \lim_{L\to\infty}\frac{1}{2L}\langle \mathsf{M}_\mu\rangle$$

$$\chi_{\mu_0,\mu_1} = \lim_{L\to\infty}\frac{\langle \mathsf{M}_{\mu_0}\mathsf{M}_{\mu_1}\rangle - \langle \mathsf{M}_{\mu_0}\rangle\langle \mathsf{M}_{\mu_1}\rangle}{2LT} =$$

$$= \lim_{L\to\infty}\frac{cov(\mathsf{M}_{\mu_0},\mathsf{M}_{\mu_1})}{2LT}$$

These quantities can be calculated by derivation of free energy:

$$m_\mu(\vec{h},T) = -\frac{\partial}{\partial h_\mu}f(\vec{h},T)$$

$$\chi_{\mu_0,\mu_1} = -\frac{\partial^2}{\partial h_{\mu_0}\partial h_{\mu_1}}f(\vec{h},T)$$

Theorem 2 (Main theorem)

For generalized double-chain three-state Potts models with Hamiltonian (1) - (2) *and zero external field* $\vec{h} = \vec{0}$ *in thermodynamic limit* (*i.e. at* $L\to\infty$) *free energy, internal energy, entropy, heat capacity, partial magnetizations and susceptibilities can be expressed through* $\lambda_{max}(T)$, *i.e. the principal eigenvalue of matrix* Θ_1:

$$f(T) = -T\frac{\ln(\lambda_{max}(T))}{2},$$

$$u(T) = T^2\frac{\partial}{\partial T}\frac{\ln(\lambda_{max}(T))}{2},$$

$$S(T) = \frac{1}{2}\left(\ln(\lambda_{max}(T)) + \frac{T}{\lambda_{max}(T)}\frac{\partial\lambda_{max}(T)}{\partial T}\right),$$

$$c(T) = 2T\frac{\partial}{\partial T}\frac{\ln(\lambda_{max}(T))}{2} + T^2\frac{\partial^2}{\partial T^2}\frac{\ln(\lambda_{max}(T))}{2},$$

$$m_\mu(\vec{h},T)\Big|_{\vec{h}=\vec{0}} = \frac{T}{2}\frac{1}{\lambda_{max}(T)}\frac{\partial\lambda_{max}(\vec{h},T)}{\partial h_\mu}\Big|_{\vec{h}=\vec{0}} = \frac{1}{3},$$

$$\chi_{\mu_0,\mu_1}(\vec{h},T)\Big|_{\vec{h}=\vec{0}} = \Big\{\frac{T}{2}\Big(\frac{1}{\lambda_{max}(T)}\frac{\partial^2\lambda_{max}(\vec{h},T)}{\partial h_{\mu_0}\partial h_{\mu_1}} -$$

$$-\frac{1}{\lambda_{max}^2(T)}\frac{\partial\lambda_{max}(\vec{h},T)}{\partial h_{\mu_0}}\frac{\partial\lambda_{max}(\vec{h},T)}{\partial h_{\mu_1}}\Big)\Big\}\Big|_{\vec{h}=\vec{0}}$$

where partial derivatives of $\lambda_{max}(\vec{h},T)$ *has the form*:

$$\frac{\partial\lambda_{max}(\vec{h},T)}{\partial h_\mu} = -\frac{\sum_{n=0}^{9}\frac{\partial a_n}{\partial h_\mu}\lambda_{max}^n}{\sum_{n=0}^{8}(n+1)a_{n+1}\lambda_{max}^n}, \quad (7)$$

$$\frac{\partial^2\lambda_{max}(\vec{h},T)}{\partial h_{\mu_0}\partial h_{\mu_1}} = \frac{-1}{\sum_{n=0}^{8}(n+1)a_{n+1}\lambda_{max}^n}\left(\sum_{n=0}^{9}\frac{\partial^2 a_n}{\partial h_{\mu_0}\partial h_{\mu_1}}\lambda_{max}^n + \right.$$

$$\frac{\partial\lambda_{max}}{\partial h_{\mu_1}}\sum_{n=0}^{8}(n+1)\frac{\partial a_{n+1}}{\partial h_{\mu_0}}\lambda_{max}^n + \frac{\partial\lambda_{max}}{\partial h_{\mu_0}}\left[\sum_{n=0}^{8}(n+1)\frac{\partial a_{n+1}}{\partial h_{\mu_1}}\lambda_{max}^n\right.$$

$$+\frac{\partial\lambda_{max}}{\partial h_{\mu_1}}\left(\sum_{n=0}^{7}(n+2)(n+1)a_{n+1}\lambda_{max}^{n}\right)\Bigg]\Bigg) \quad (8)$$

A. Commentary

In this paper, thermodynamic characteristics were found only at zero external field. It is clear that in this case (see (3)) partial magnetizations $m_\mu = \frac{1}{3}$, because $m_{\mu_1} = m_{\mu_2}$ and $\sum_{\mu=0}^{2} m_\mu = 1.$

Formulae (7), (8) are obtained by taking the derivative of the transfer-matrix's Θ secular equation: $\sum_{n=0}^{9} a_n\lambda^n = 0$ with respect to h_μ. In a similar way, one can differentiate secular equation with respect to T and obtain analogous formulae for entropy, internal energy and heat capacity as is done in [40].

III. Generalized clock Potts model at $q = 3$

Let us rewrite the elementary Hamiltonian (2) with zero external field $\vec{h} = \vec{0}$ in the form of generalized clock Potts model's Hamiltonian:

$$H(\sigma_0,\sigma_1;\sigma_2,\sigma_3) = -\Big[A_1\cos\Big(\frac{2\pi}{3}(\sigma_1-\sigma_0)\Big)+A_2\cos\Big(\frac{2\pi}{3}(\sigma_2-\sigma_0)\Big)+$$
$$A_3\cos\Big(\frac{2\pi}{3}(\sigma_3-\sigma_0)\Big)+A_4\cos\Big(\frac{2\pi}{3}(\sigma_3-\sigma_1)\Big)+A_5\cos\Big(\frac{2\pi}{3}(\sigma_2-\sigma_1)\Big)+$$
$$A_6\cos\Big(\frac{2\pi}{3}(\sigma_3-\sigma_2)\Big)+\tilde{A}_1\sin\Big(\frac{2\pi}{3}(\sigma_1-\sigma_0)\Big)+\tilde{A}_2\sin\Big(\frac{2\pi}{3}(\sigma_2-\sigma_0)\Big)+$$
$$\tilde{A}_6\sin\Big(\frac{2\pi}{3}(\sigma_3-\sigma_2)\Big)+B_1\cos\Big(\frac{2\pi}{3}(\sigma_0+\sigma_1+\sigma_2)\Big)+B_2\cos\Big(\frac{2\pi}{3}(\sigma_0+\sigma_1+\sigma_3)\Big)+$$
$$B_3\cos\Big(\frac{2\pi}{3}(\sigma_0+\sigma_2+\sigma_3)\Big)+B_4\cos\Big(\frac{2\pi}{3}(\sigma_1+\sigma_2+\sigma_3)\Big)+\tilde{B}_1\sin\Big(\frac{2\pi}{3}(\sigma_0+\sigma_1+\sigma_2)\Big)$$
$$+\tilde{B}_2\sin\Big(\frac{2\pi}{3}(\sigma_0+\sigma_1+\sigma_3)\Big)+\tilde{B}_3\sin\Big(\frac{2\pi}{3}(\sigma_0+\sigma_2+\sigma_3)\Big)+\tilde{B}_4\sin\Big(\frac{2\pi}{3}(\sigma_1+\sigma_2+\sigma_3)\Big)$$
$$+C_1\cos\Big(\frac{2\pi}{3}(\sigma_0-\sigma_1+\sigma_2-\sigma_3)\Big)+C_2\cos\Big(\frac{2\pi}{3}(\sigma_0-\sigma_1-\sigma_2+\sigma_3)\Big)+$$
$$C_3\cos\Big(\frac{2\pi}{3}(\sigma_0+\sigma_1-\sigma_2-\sigma_3)\Big)+C_4+\tilde{C}_1\sin\Big(\frac{2\pi}{3}(\sigma_0-\sigma_1+\sigma_2-\sigma_3)\Big)+$$
$$\tilde{C}_2\sin\Big(\frac{2\pi}{3}(\sigma_0-\sigma_1-\sigma_2+\sigma_3)\Big)+\tilde{C}_3\sin\Big(\frac{2\pi}{3}(\sigma_0+\sigma_1-\sigma_2-\sigma_3)\Big)\Big] \quad (9)$$

Hamiltonian (9) is equivalent to the original one (2) with $\vec{h} = \vec{0}$, as both of them cover all functions on X^4 (where $X = \{0,1,2\}$) which are invariant under cyclic shift of all spin values. That is why results of section II are valid for such clock models. The main advantage of clock version of Hamiltonian is that in this form double, triple and quadruple interactions are extracted.

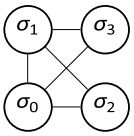


Fig. 3. Step of the transfer-matrix Θ, only two-site couplings are displayed.

It should be marked that in further calculations constant C_4 is set equal to zero, moreover, transfer-matrix Θ, propagating from left to right, will cover two-site interaction of σ_2 and σ_3 in the next step (see Figure 3), that is why we put: $A_6 = \tilde{A}_6 = C_4 = 0$

IV. Examples of physical characteristics

In order to prevent increasing the number of parameters, constants of double, triple and quadruple forces are usually set equal to each other (see [27], [32]). Let us put in Hamiltonian [9]: $A_i \equiv A, \tilde{A}_i \equiv \tilde{A}, i = 1,\ldots,5;$

$B_j \equiv B, \tilde{B}_j \equiv \tilde{B}, j = 1,\ldots,4; C_k \equiv C, \tilde{C}_k \equiv \tilde{C}, k = 1,2,3.$

We will change only double interactions:

$A = \tilde{A} = r,\ r \in [-1,1]$ and set $B = 0, C = 0.1, \tilde{B} = 0.2, \tilde{C} = 0.3.$

Let us show the free energy graph of this model:

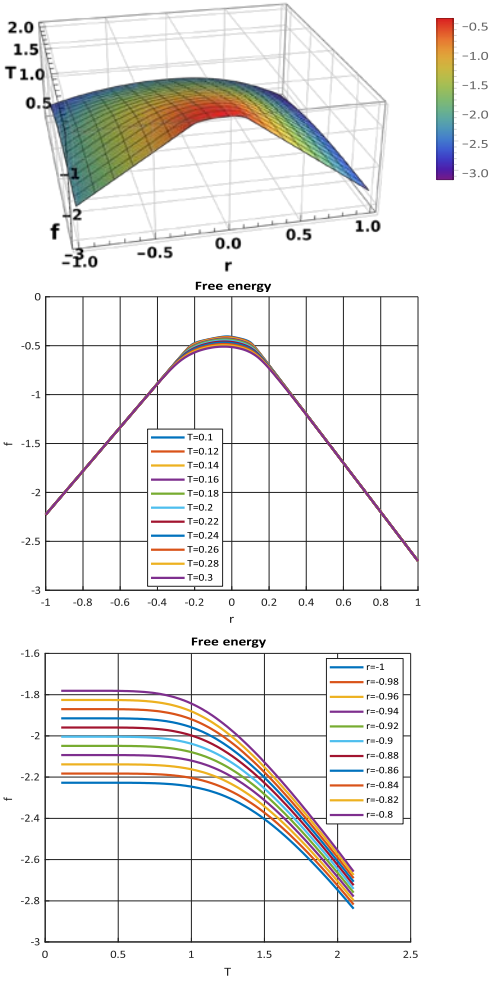


Fig. 4. Plots of free energy, $r \in [-1,1], T \in [-0.1; 2.1]$

The most interesting is model's behavior at low temperature, i.e. at $T \to 0+$. It is obvious that $\lim_{T\to 0+} f(r,T) = \lim_{T\to 0+} u(r,T)$. In addition (see table 2), one can easily show that at $T \to 0+$ after replacing the matrix's Θ_1 secular equation coefficients by equivalent infinitely large ones λ_{max} will satisfy:

$$\lambda^3 - e^{-\frac{2u_1}{T}}\lambda^2 - e^{-\frac{6u_2}{T}} = 0 \quad at \quad r \in (-\infty; r_2),$$

$$\lambda^3 - e^{-\frac{6u_2}{T}} - e^{-\frac{6u_3}{T}} = 0 \quad at \quad r \in (r_1; r_3),$$

$$\lambda^3 - e^{-\frac{2u_4}{T}}\lambda^2 - e^{-\frac{6u_3}{T}} = 0 \quad at \quad r \in (r_2; \infty),$$

where $u_i, i = 1,\ldots,4$ denote the limits of internal energy at $T \to 0+$ (see subsection IV.A).

A. *Structure of ground states*

Transfer-matrix (4) can be interpreted as a weight matrix of some directed graph with 9 vertices. Obviously, the ground state (i.e. the state with minimal energy per spin) must be periodic, since the number of vertices is finite, and therefore the initial vertex must be repeated at some step $L \le 9$. However, the number of paths of length L starting and ending at the same vertex is finite, hence there exists a path (and its corresponding spin configuration σ) with minimal energy per spin: $\frac{\mathsf{H}_L(\sigma)}{2L} \to min$. Thus, the search of ground states can be reduced to the search of a minimum mean weight cycle in a directed graph. It is clear that such a cycle can be started from any of its elements (compare, for instance, spin configurations at $r \in (r_1; r_2)$). Moreover (see (3)), states derived from ground ones by cyclic shift of all spin values are also ground states. That is why only one of them is presented (see table 2):

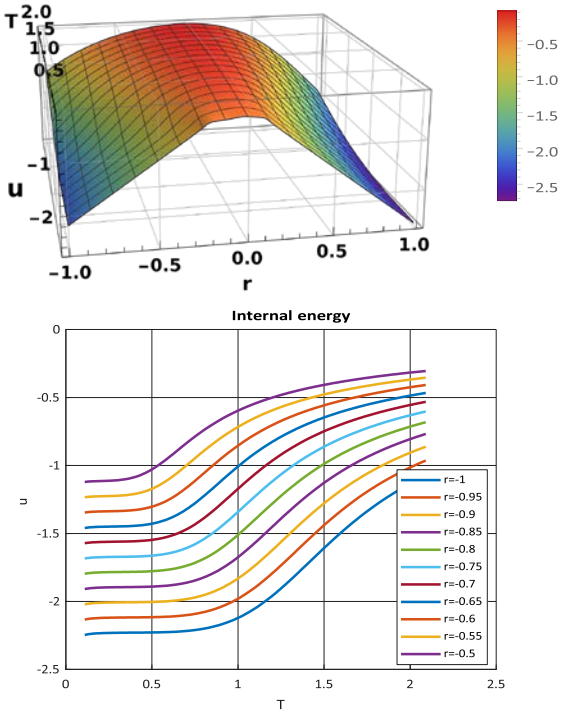
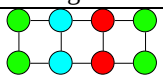
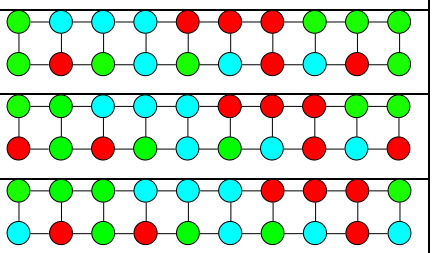
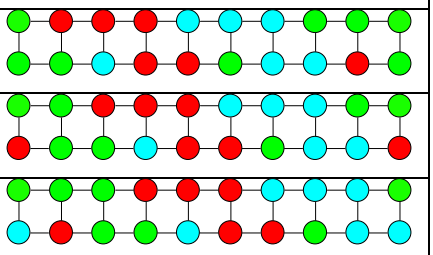


Fig. 5. Plots of internal energy, $r \in [-1,1], T \in [-0.1; 2.1]$

Table 1. Color designation.

Color			
State	0	1	2

Table 2. Ground states.

Value of r	Configuration σ of ground state
$r \in \left(-\infty; r_1 = \frac{3(7-13\sqrt{3})}{10(11\sqrt{3}+3)}\right)$	
$r \in \left(r_1; r_2 = \frac{3(1-\sqrt{3})}{20(\sqrt{3}+3)}\right)$	
$r \in \left(r_2; r_3 = \frac{33(1-\sqrt{3})}{10(\sqrt{3}-27)}\right)$	
$r \in (r_3; \infty)$	

Also ground states at r_i are ignored to avoid making the table 2 more cumbersome.

As internal energy is the average of Hamiltonian per spin, we obtain that $\lim\limits_{T\to 0+} u(r,T) = \frac{\mathsf{H}_L(\sigma)}{2L}$, herewith configuration σ can be found in table 2:

$$\lim_{T\to 0+} u(r,T) = \begin{cases} u_1 = \left(\sqrt{3}+\frac{1}{2}\right)r + \frac{1}{40}\left(3\sqrt{3}-5\right), r \in [-\infty; r_1 = \frac{3(7-13\sqrt{3})}{10(11\sqrt{3}+3)} \approx -0.211] \\ u_2 = \frac{1}{60}\left(5\left(\sqrt{3}+3\right)r - 15\sqrt{3}+3\right), r \in [r_1; r_2 = \frac{3(1-\sqrt{3})}{20(\sqrt{3}+3)} \approx -0.023] \\ u_3 = \frac{1}{120}\left(-10\left(\sqrt{3}+3\right)r - 33\sqrt{3}+9\right), r \in [r_2; r_3 = \frac{33(1-\sqrt{3})}{10(\sqrt{3}-27)} \approx 0.096] \\ u_4 = -\frac{5r}{2} - \frac{1}{5}, \qquad r \in [r_3; \infty] \end{cases} \tag{10}$$

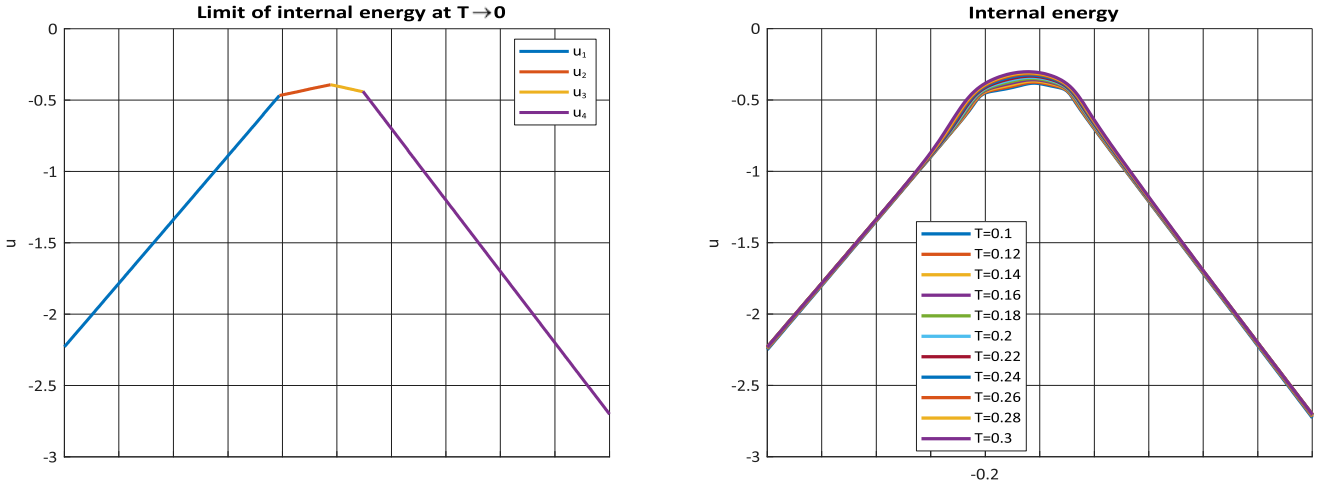


Fig. 6. Plots of internal energy in low-temperature region and its limit at $T \to 0$, $r \in [-1,1]$

One can plot this function (10) and compare it with the plot of internal energy at low temperature (see Figure 6).

B. Entropy

At low temperature entropy $S \approx \frac{\ln N'}{2L}$, where L – length of double chain, N' – number of configurations with minimal energy (i.e. configurations that survive at $T \to 0+$) and therefore the limit $\lim\limits_{T\to 0+} S(r,T)$ is equal to zero everywhere, except of three frustration points r_i, since the number of ground states at other points is finite (see table 2). At the first and third frustration points, the limit is equal to half of the logarithm of equation's $\lambda^3 - \lambda^2 - 1 = 0$ maximum root, that is

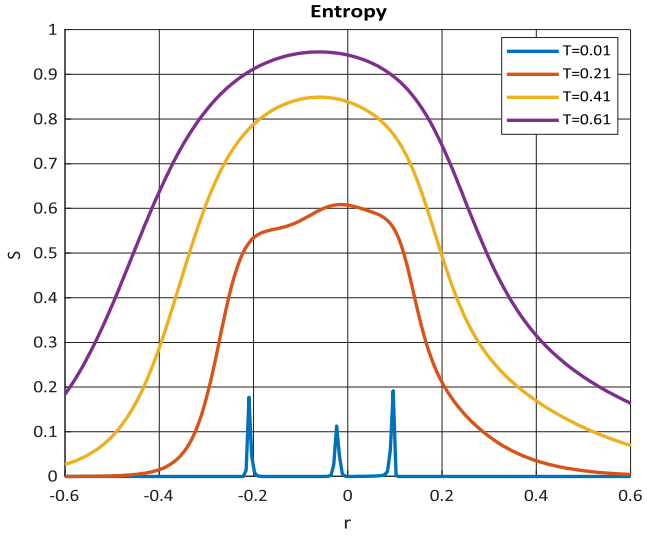
$$\lim_{T\to 0+} S(r,T) = \frac{1}{2}\ln\left(\frac{1}{3}\left(\sqrt[3]{\frac{1}{2}\left(3\sqrt{93}+29\right)}+\sqrt[3]{\frac{1}{2}\left(29-3\sqrt{93}\right)}+1\right)\right) \approx 0.1911$$

at $r = r_i, i = 1,3$.

At the second frustration point

$$\lim_{T\to 0+} S(r_2,T) = \frac{1}{6}\ln 2 \approx 0.1155$$

One can see it on the plot (see Figure 7):



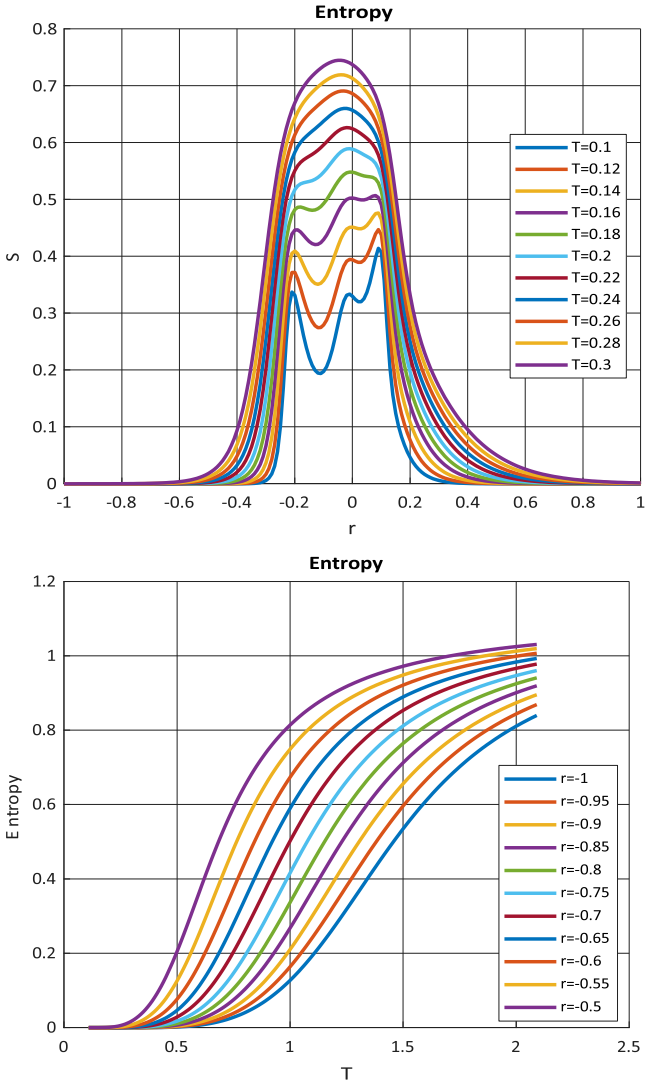


Fig. 7. Plots of entropy, $r \in [-1,1], T \in [0.1;2.1]$

Clearly $\lim\limits_{T\to\infty} S(r,T) = \ln 3 \approx 1,099$, because at high temperature all 9^L configurations have similar energy.

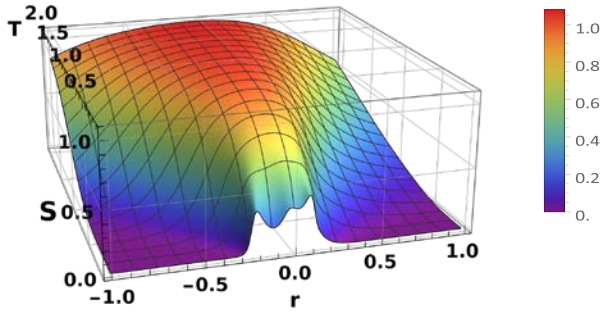


Fig. 8. Plot of entropy, $r \in [-1,1], T \in [0.1;2.1]$

Let us show the plot of heat capacity. In the low-temperature region (9) one can see three sharp minima at frustration points:

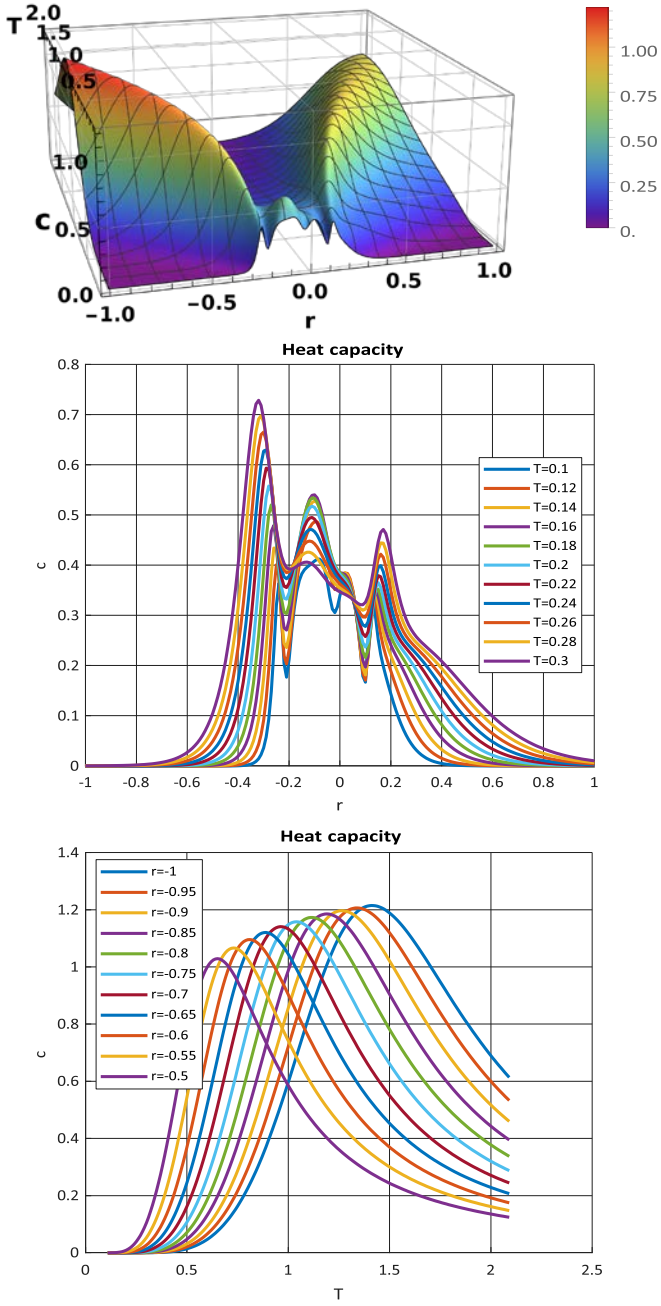


Fig. 9. Plots of heat capacity, $r \in [-1;1], T \in [0.1;2.1]$

As for partial susceptibilities, we should note that $\chi_{\mu,\mu} \geq 0$, however $\chi_{\mu,\nu}$ with $\mu \neq \nu$ can be negative. Mark that in the low-temperature region $\chi_{0,1}$ suffers jumps at frustration points (see Figure 10):

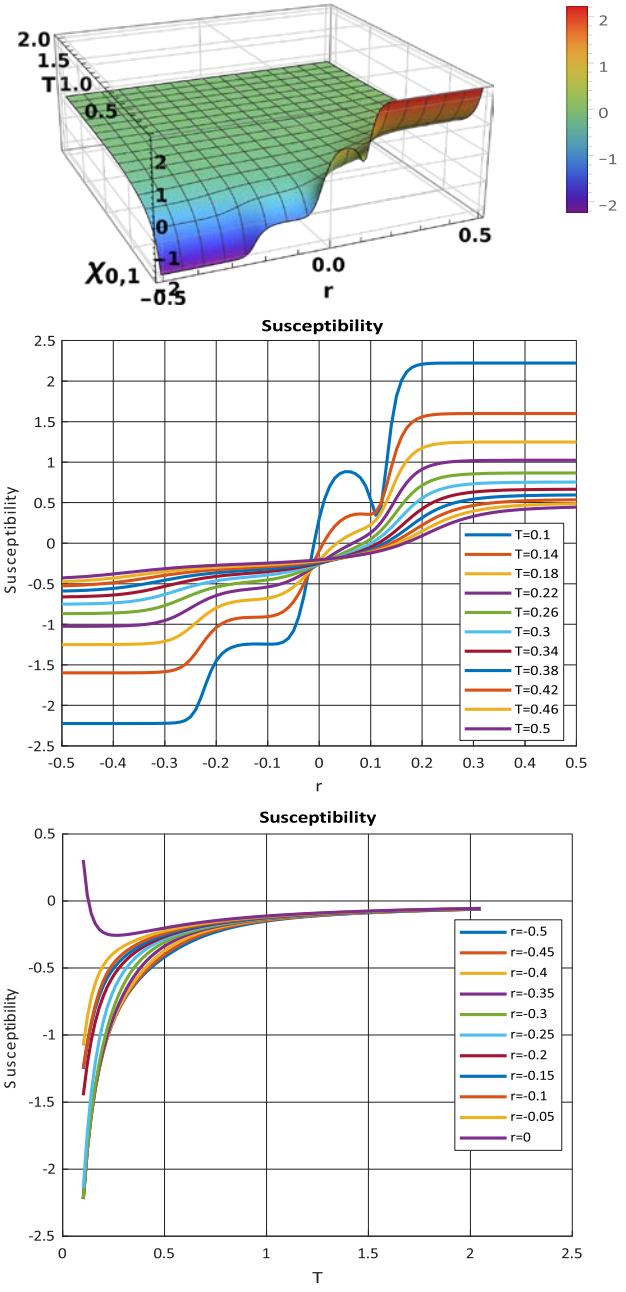


Fig. 10. Plots of $\chi_{0,1}$, $r \in [-0.5;0.5], T \in [0.1;2.1]$

V. Proof of Theorems

It is known that commuting matrices conserve eigensubspaces of each other [36]: $AB = BA \Rightarrow AV_\lambda \subset V_\lambda$,

where $V_\lambda -$ eigensubspace of matrix B. Indeed, if $AB = BA$ and x is an eigenvector of matrix B, corresponding to eigenvalue λ, then vector Ax is also an eigenvector of B, corresponding to eigenvalue λ. In other words, eigensubspaces of one operator must be invariant subspaces of another one.

Let us formulate famous result from linear algebra: if linear space V can be decomposed into a direct sum of invariant subspaces of Θ : $\mathsf{V} = \mathsf{W} \oplus \mathsf{Y} \oplus \ldots$, then

spectrum of Θ is equal to the union of spectra of Θ's restrictions to its invariant subspaces: $\sigma(\Theta) = \sigma(\Theta|_W) \bigcup \sigma(\Theta|_Y) \bigcup \ldots$. Since generalized eigenvectors of Θ belongs to its invariant subspace, this means that search for generalized eigenvectors vectors of Θ can be performed not on the whole space, but on invariant subspaces, which are easy to find, because they are eigensubspaces of permutation matrix Щ (see (11)). Finally, we obtain the form of the generalized eigenvectors:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \varepsilon v_3 \\ \varepsilon v_1 \\ \varepsilon v_2 \\ \varepsilon^2 v_2 \\ \varepsilon^2 v_3 \\ \varepsilon^2 v_1 \end{pmatrix}$$

where ε is a cubic root of 1. Hence (according to the Perron-Frobenius theorem, the eigenvector of matrix with positive elements corresponding to the principle eigenvalue have positive coordinates) the principal eigenvalue of the transfer-matrix Θ coincides with the one of Θ_1.

A. *The form of the eigenvectors*

Permutation matrices permute the components of a vector. This fact allows us to find eigenvectors of these matrices [35]. A permutation can be represented as a product of non-intersecting cycles. Let a permutation consist of only one cycle, for instance, (123) . Then vectors of the form $\left(1, \varepsilon, \varepsilon^2\right)^T$, where $\varepsilon = \sqrt[3]{1}$, are eigenvectors corresponding to the eigenvalue ε . When permutation is a product of several cycles one can easily find eigenvectors in a similar manner.

For example, matrix Щ (5) corresponds to permutation $(159)(267)(348)$. Eigenvectors of matrix Щ , corresponding to eigenvalue λ , have the following form:

$$\left.\begin{array}{l} (1,0,0,0,\varepsilon,0,0,0,\varepsilon^2)^T \\ (0,1,0,0,0,\varepsilon,\varepsilon^2,0,0)^T \\ (0,0,1,\varepsilon,0,0,0,\varepsilon^2,0)^T \end{array}\right\} \lambda = \varepsilon \qquad (11)$$

VI. Conclusion

In this paper, the new type of Potts model is introduced, which generalizes clock and standard models. In the clock interpretation, this model has 24 forces in a unit cell: 10 double, 8 triple and 6 quadruple ones. Suggested model covers all double-chain three-state Potts models with multi-site interactions which are invariant under cyclic shift of all spin values. An exact analytical solution of this model is obtained, formulae for its physical characteristics are given. Using the commutation of the transfer-matrix Θ with permutation matrix, the search of 9×9 matrix's spectrum is reduced to the search of spectra of three matrices 3×3 with some specific structure. Also some generalized characteristics such as partial magnetization and susceptibility are introduced. An example of the model is analyzed in detail: showed its ground states' structure and behavior at low temperature.

VII. Appendix

A. *The Cardano formula*

To solve cubic equation $x^3 + ax^2 + bx + c = 0$ at first one need (with the help of substitution $x + \frac{a}{3} = y$) to eliminate monomial ax^2 and get $y^3 = -py - q$. Then if in $(d+e)^3 = 3de(d+e) + d^3 + e^3$, put $d + e = y$, we obtain a system:

$$\begin{cases} -q = d^3 + e^3 \\ -p = 3de \end{cases} \leftrightarrow \begin{cases} -q = d^3 + e^3 \\ -\dfrac{p^3}{27} = d^3 e^3 \end{cases}$$

The last system can be easily solved. Let the pair of numbers (d_0, e_0) be its solution. Then $(d_0\varepsilon^2, e_0\varepsilon), (d_0\varepsilon, e_0\varepsilon^2)$ are also its solutions (ε is some complex cubic root of 1). Eventually, we get the Cardano formula:

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$$

References

[1] Niss, M. History of the Lenz-Ising Model 1920–1950: From Ferromagnetic to Cooperative Phenomena. Arch. Hist. Exact Sci., **59**, 267–318 (2005). DOI: 10.1007/s00407-004-0088-3

[2] T. D. Silva, Vattika Sivised, A statistical mechanics perspective for protein folding from q-state Potts model. arXiv:1709.04813

[3] Murata, K. K. Hamiltonian formulation of site percolation in a lattice gas. *J. of Phys. A: Math. Gen.* **12(1)**, 81 (1979). DOI 10.1088/0305-4470/12/1/020

[4] Coniglio, A., Stanley, H. E., Klein, W. Site-bond correlated-percolation problem: a statistical mechanical model of polymer gelation. *Phys. Rev. Lett.* **42(8)**, 518 (1979). DOI: 10.1103/PhysRevLett.42.518

[5] L. Sun, Y.F.Chang and X.Cai, A discrete simulation of tumor growth concerning nutrient influence. International Journal of Modern Physics B, vol. 18, Nos. 17–19 (2004) 2651–2657. DOI: 10.1142/S0217979204025853

[6] Utkir A. Rozikov, Gibbs Measures in Biology and Physics. The Potts Model. *World Scientific* (2023). DOI: 10.1142/12694

[7] L. Beaudin, A review of the Potts model. Rose-Hulman Undergraduate Math. Jour. 8(1) (2007)

[8] H. A. Kramers, G. H. Wannier, Statistics of the Two-Dimensional Ferromagnet. Part I. *Phys. Rev.* **60**, 252 (1941). DOI: 10.1103/PhysRev.60.252

[9] P.C.Hemmer, H.Holden, S.K.Ratkje, The collected works of Lars Onsager (With Commentary). *World Scientific*(1996). DOI: 10.1142/3027

[10] R. B. Potts, Some generalized order-disorder transformations. *Proc. Camb. Phil. Soc.*, **48** (1952) , pp. 106 - 109. DOI: 10.1017/S0305004100027419

[11] F.A. Kassan-Ogly, One-dimensional 3-state and 4-state standard Potts models in magnetic field. *Phase*

Transitions, Vol. 71 , pp. 39-55. DOI:10.1080/01411590008228976

[12] Kassan-Ogly, F.A., Filippov, B.N., Proshkin, A.I. et al. Twelve-state Potts model in a magnetic field. *Phys. Metals Metallogr.*, **116**, 115–127 (2015), DOI: 10.1134/S0031918X15020088

[13] F.A. Kassan-Ogly, Modified 6-state and 8-state Potts models in magnetic field. *Phase Transitions*, Vol. 72, (2000) pp. 223-231, DOI: 10.1080/01411590008228991

[14] M.A.Yurishchev, Double Potts chain and exact results for some two-dimensional spin models. *JETP*, Volume 93, pages 1113–1118, (2001) . DOI: 10.1134/1.142718344

[15] Khrapov P. V. Disorder Solutions for Generalized Ising and Potts Models with Multispin Interaction, *Sovremennye informacionnye tehnologii i IT-obrazovanie = Modern Information Technologies and IT-Education*, **15(1)** (2019), 33-44. DOI: 10.25559/SITITO.15.201901.33-44

[16] Joanna A. Ellis-Monaghan, Iain Moffatt, Handbook of the Tutte Polynomial and Related Topics. *Chapman & Hall* (2024)

[17] A.D. Sokal, The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In: Surveys in Combinatorics, edited by Bridget S. Webb, *Cambridge University Press*, 2005, pp. 173-226. DOI: 10.1017/CBO9780511734885.009

[18] C.M. Fortuin, P.W. Kasteleyn, On the random-cluster model: I. Introduction and relation to other models. *Physica*, **57** (1972), 536-564, DOI: 10.1016/0031-8914(72)90045-6

[19] R. A. Minlos, P. V. Khrapov, Cluster properties and bound states of the Yang–Mills model with compact gauge group. I. *Theoret. and Math. Phys.*, **61**, 1261–1265 (1984). DOI: 10.1007/BF01035013

[20] F.A. Kassan-Ogly, B.N. Filippov, Correlation function in a one-dimensional 4-state standard Potts model with next-nearest-neighbor interaction. *The Physics of Metals and Metallography*, **95**(6):518-530 (2003).

[21] K.P. Kalinin and N.G. Berloff, Simulating Ising and Potts models and external fields with non-equilibrium condensates. *Phys. Rev. Lett.*, **121**, 235302 (2018). DOI: 10.1103/PhysRevLett.121.235302

[22] Andreev A. S., Khrapov P. V. Simulating Ising-like models on quantum computer. Sovremennye informacionnye tehnologii i IT-obrazovanie= Modern Information Technologies and IT-Education. – 2023. – Vol. 19. – №. 3.
DOI: 10.25559/SITITO.019.202303.554-563

[23] N.S. Ananikian, R.G. Ghulghazaryan, Exact solution of a Z(4) gauge Potts model on planar lattices, *Phys. Rev. E* **60**, 5106 (1999), DOI: 10.1103/PhysRevE.60.5106

[24] R.G.Ghulghazaryan, N.S.Ananikian, Partition function zeros of the one-dimensional Potts model: The recursive method. *J. Phys. A: Math. Gen.* **36**, 6297 (2003). DOI: 10.1088/0305-4470/36/23/302

[25] R.G. Ghulghazaryan, N.S. Ananikian, P.M.A. Sloot, Yang-Lee zeros of the Q-state Potts model on recursive lattices. *Phys. Rev. E* **66**, 046110 (2002). DOI: 10.1103/PhysRevE.66.046110

[26] Alan D. Sokal, Bounds on the Complex Zeros of (Di)Chromatic Polynomials and Potts-Model Partition Function, *Comb.Probab.Comput.*, **10** (2001), 41-77. DOI: 10.1017/S0963548300004612

[27] Shu-Chiuan Chang, Robert Shrock, Exact Partition Function for the Potts Model with Next-Nearest Neighbor Couplings on Strips of the Square Lattice. *Int. J. Mod. Phys. B15*, vol. 15, No. 5, 443-478 (2001). DOI: 10.1142/S0217979201004630

[28] S.C. Chang, R. Shrock, Some exact results on the Potts model partition function in a magnetic field. *J.Phys. A 42* (2009), no. 38, 385004, 5 pp. DOI: 10.1088/1751-8113/42/38/385004

[29] S.C. Chang, R. Shrock, Weighted Graph Colorings, *J. Stat. Phys.*, **138**, (2010) 496-542. DOI: 10.1007/s10955-009-9882-2

[30] Shu-Chiuan Chang, Robert Shrock, Ground State Entropy of the Potts Antiferromagnet with Next-Nearest-Neighbor Spin-Spin Couplings on Strips of the Square Lattice. *Phys. Rev. E*, **62**, 4650 (2000), DOI: 10.1103/PhysRevE.62.4650

[31] Shu-Chiuan Chang, Robert Shrock, Exact Potts Model Partition Functions for Strips of the Honeycomb Lattice. *J. Stat. Phys.*, **130**, 1011–1024 (2008). DOI: 10.1007/s10955-007-9461-3

[32] Wu, F. Y. The Potts model. *Reviews of Modern Physics*, **54(1)** (1982), 235–268. DOI: 10.1103/revmodphys.54.235

[33] Khrapov, P.V. Cluster expansion and spectrum of the transfer matrix of the two-dimensional Ising model with strong external field. *Theor. Math. Phys.*, **60**,734–735 (1984), DOI:10.1007/BF01018259

[34] Kashapov, I. A. Structure of ground states in three-dimensional using model with three-step interaction. Teoreticheskaya i Matematicheskaya Fizika 33.1 (1977): 110-118, DOI:10.1007/bf01039015

[35] M. Isabel Garca-Planas, M. Dolors Magret, Eigenvectors of Permutation Matrices. *Advances in Pure Mathematics*, **5** (2015), 390-394. DOI:10.4236/apm.2015.57038

[36] Horn, R. A., Johnson, C. R., Matrix analysis. *Cambridge university press* (2012).

[37] Cardano G., Witmer T. R., Ore O., The rules of algebra: Ars Magna. *Courier Corporation*, **685** (2007).

[38] Tabachnikov S.L., Fuks D.B., Mathematical divertissement. *MCNMO*, (2011).

[39] Baxter R.J. Exactly solved models in statistical mechanics. *Elsevier*, (2016).

[40] P. Khrapov, N. Volkov, Exact solution of generalized triple Ising chains with multi-spin interactions. arXiv:2406.10683

Точное решение двухцепочечной обобщенной модели Поттса с тремя состояниями

П. В. Храпов, Г. А. Скворцов

***Аннотация* — Получено точное решение двухцепочечной циклически замкнутой решеточной обобщенной модели Поттса с мультиспиновыми взаимодействиями со спином, принимающим три значения и произвольным гамильтонианом, инвариантным относительно циклического сдвига всех значений спинов. В полосе конечной длины найдена статистическая сумма и свободная энергия. В термодинамическом пределе получены также и другие физические характеристики модели: внутренняя энергия, энтропия и теплоемкость. Определены и найдены парциальные намагниченности и парциальные восприимчивости, как обобщение намагниченности и восприимчивости. Результаты получены как для моделей с гамильтонианами, выраженными через символы Кронекера, так и для обобщенных моделей с гамильтонианами типа clock. Подробно рассмотрен пример со всеми 24 параметрами взаимодействия всех возможных типов, приведены графики термодинамических характеристик, найдена структура основных состояний при некоторых константах взаимодействия.**

***Ключевые слова* — модель Поттса, модель Изинга, решетка, гамильтониан, трансфер-матрица, точное решение, статистическая сумма, свободная энергия, внутренняя энергия, теплоемкость, намагниченность, восприимчивость.**

Об авторах:

Храпов Павел Васильевич,
доцент кафедры «Высшая математика», факультет «Фундаментальные науки», ФГБОУ ВО «Московский государственный технический университет имени Н.Э. Баумана (национальный исследовательский университет)» (105005, Российская Федерация, г. Москва, ул. 2-я Бауманская, д. 5, к. 1), кандидат физико-математических наук,
ORCID: https://orcid.org/0000-0002-6269-0727,
khrapov@bmstu.ru, pvkhrapov@gmail.com

Скворцов Григорий Арсеньевич,
студент кафедры «Высшая математика», факультет «Фундаментальные науки», ФГБОУ ВО «Московский государственный технический университет имени Н.Э. Баумана (национальный исследовательский университет)» (105005, Российская Федерация, г. Москва, ул. 2-я Бауманская, д. 5, к. 1),
ORCID: https://orcid.org/0009-0005-5140-785X,
skvortsovga@student.bmstu.ru , skvortsov.grisha@list.ru