A Solution Approach to the Valuation of Contingent Claim by Utility Indifference with Unpredictable Risks

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Abstract — **In this article, we consider a variant of the Merton problem, where we choose a stochastic process (a risky tradable asset or a risky commodity asset) with random drift and volatility. It is well known in the literature that this leads to a stochastic optimal control problem, which allows for the derivation of a parabolic partial differential equation, known as the Hamilton-Jacobi-Bellman (HJB) equation. This equation is commonly used to evaluate certain credit instruments, such as corporate bonds and credit default swaps (CDS), which is the case in our present work. The construction or definition of the value function involves a power transformation based on the solution of a linear or semi linear parabolic equation. We will use reduced solutions of these equations to determine prices of corporate bonds and credit default swap spreads under utility indifference. This approach will not only allow us to obtain analytical results, but also provide better insights into the dynamics of credit markets. The derived formulas for the valuation of corporate bonds and credit default swaps (CDS) can be used to develop trading strategies in credit markets. Furthermore, the prediction of credit spread dynamics and default risk will enable investors to construct complex trading portfolios, such as arbitrage and hedging strategies.**

Keywords— **Stochastic control, Utility indifference, Value function, Viscosity solution.**

I. Introduction

In the field of financial mathematics, the theory of stochastic optimal control plays an important role, especially in the solution of optimal investment problems initiated by Robert Merton. In the article [1], Merton presents the optimal portfolio investment problem, which was later generalized by research introducing more realistic underlying asset dynamics [2] and [3].

In this work, we consider a variant of the Merton investment problem with stochastic drift and volatility. In general, and in particular in [2], such a problem under certain assumptions about the underlying process and the utility function leads to a parabolic PDE of the linear [14] or non-linear type, whose solutions can, for example in mathematical finance be used to determine the price of certain financial instruments.

One of the important problems in mathematical finance is the study related to the valuation of contingent claims in an incomplete market [13].

In this paper, we refer to the work of Georges Sigloch [4] and consider the following situation: An investor makes a loan (a debt security) to a bond issuer. During the term of the contract, the bond issuer may default (fail to pay principal or interest), leading the investor to seek protection against the risk of default from a credit protection seller. This raises a number of questions, in particular: firstly, how to evaluate the price of the financial contract (corporate bond) that will be signed between the investor and the bond issuer and secondly, how to evaluate the price of the protection contract (CDS) that will be signed between the investor and the credit protection seller.

Proposed answers to these questions, in the case of a reduced-form model, were developed in [4] for constant underlying parameters (drift and volatility). In this paper, we propose a generalization of some of the results of [4], by determining the indifference prices signed between the investor and the bond issuer, then between the investor and the credit protection seller, when the underlying parameters are stochastic, inspired by the work of [2].

The paper is organized as follows: in section 2, we present the basic hypotheses of our model and we introduce the Hamilton-Jacobi-Bellman equation associated with the problem; in section 3, we provide the corporate Bond price and CDS spread by utility indifference method in the case of a underlying with stochastic drift and volatility; in section 4, we present the definition of viscosity solutions of non-linear parabolic partial differential equations and some simulation curves of the paper.

II. Setup of the model

We consider an investor who has at time t a self-financing portfolio with non-risky asset (bond) M_t subject

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to an interest rate function depending on time r_t and tradable risky assets (stock account) without default A_t^1.
The dynamics of these non-risky and risky assets are respectively given by the equations:

$$\forall t \in [0,T], dM_t = r_t M_t dt, M_0 = m \quad (1)$$

$$\forall t \in [0,T], dA_t^1 = \mu_t A_t^1 dt + \sigma_t A_t^1 dB_t, A_0^1 = a^1 \quad (2)$$

The process $(Z_t)_{t\in[0,T]}$, will be referred to as the stochastic factor [2] and satisfying:

$$dZ_t = \alpha_t dt + \beta_t dW_t \quad (3)$$

such that $Z_0 = z_0 \in R$.
The functions $\mu_t = \mu(t, Z_t), \sigma_t = \sigma(t, Z_t), \alpha_t = \alpha(t, Z_t)$ and $\beta_t = \beta(t, Z_t)$ are assumed to satisfy all the regularity assumptions required to guarantee the existence and uniqueness of the solution of each of equations (2) and (3).
Using concepts from stochastic theory, we specify these basic assumptions about market coefficients: The functions $\mu, \sigma, \alpha, \beta$: $R \times [0,T] \rightarrow R$ satisfy the global Lipschitz and linear growth conditions:

$$|f(t,z) - f(t,\bar{z})| \leq k|z - \bar{z}| \quad (4)$$

$$f^2(t,z) \leq k^2(1 + z^2) \quad (5)$$

for each $\forall t \in [0,T], z, \bar{z} \in R$ and k being a positive constant and f representing μ, σ, α and β. Such that the conditions (4) and (5) are standard for the existence and uniqueness of solutions of the state stochastic differential equations (2) and (3) (see (5)).
On the probability space $(\Omega, \mathcal{F}, P)$, the Brownian motions B_t and W_t are correlated with coefficient correlation $\rho \in [-1,1]$.
At any time $s \in [t,T]$, it is reasonable to assume that the investor has complete information about the price of the risky asset A_s^1. We model the state of the information given to the investor by $\mathcal{F}_s = \sigma(\{B_u, W_u : t \leq u \leq s\} \cup \mathcal{N})$ with $\mathcal{N}$ the set of negligible subsets of Ω and $(\mathcal{F}_s)_{s\in[t,T]}$ satisfies the usual conditions: it is complete (contains $\mathcal{N}$), continuous on the right and increasing.
We define π_s^0 as the amount invested in M_s and π_s as the amount invested in A_s^1. The control π_s is assumed to be admissible, i.e.: satisfies the integrability condition $E\left[\int_t^T \sigma_s^2 \pi_s^2 ds\right] < +\infty$ and is $\mathcal{F}_s$-progressively measurable. The total wealth of the investor satisfies the budget constraint $\Lambda_s = \pi_s^0 + \pi_s$ and by the hypothesis of self-financing of the portfolio, its dynamic is defined by SDE:

$$d\Lambda_s = [r_s \Lambda_s + \pi_s(\mu_s - r_s)]\, ds + \pi_s \sigma_s dB_s \quad (6)$$

with $\Lambda_t = \lambda \geq 0, t \leq s \leq T$ and as wealth, at any moment s it must be positive almost everywhere. The value function of the investor is

$$\Psi(t,\lambda,z) = \sup_{\pi_t \in \mathcal{A}} E[u(\Lambda_T) | \Lambda_t = \lambda, Z_t = z] \quad (7)$$

where $\mathcal{A}$ is the set of admissible policies, Λ_T is the terminal wealth and u is a *CARA* (Constant Absolute Risk Aversion) utility function that is concave and non-decreasing

$$u(x) = -e^{-\gamma x}, \ \gamma > 0 \quad (8)$$

where γ the risk aversion coefficient.
The representation of the value function in a separable form $\Psi(t,\lambda,z) = u\left(\lambda e^{\int_t^T r_s ds}\right) g(t,z)$ allows to highlight a function g in general unknown and which verifies a nonlinear partial differential equation. Using the reasoning of [2], we transform the function g into a power of another function which will be the solution of a reduced parabolic linear equation and in another case a semi-linear equation.
Under exponential utility function u written on the wealth of the investor, the value function is represented by $\Psi(t,\lambda,z) = -e^{-\gamma\lambda e^{\int_t^T r_s ds}} g(t,z)$ where $g: [0,T] \times R \rightarrow R^+$ verifies the nonlinear parabolic equation:

$$\begin{cases} g_t + \frac{\beta_t^2}{2} g_{zz} + \left[\alpha_t - \frac{\rho\, \beta_t(\mu_t - r_t)}{\sigma_t}\right] g_z - \frac{1}{2}\left(\frac{\mu_t - r_t}{\sigma_t}\right)^2 g - \frac{\rho^2 \beta_t^2}{2} \frac{g_z^2}{g} = 0 \\ g(T,z) = 1 \end{cases}$$
(9)

such that the proof of (9) can be found in the APPENDIX in *Proof 1* in the subsection *Some Proofs*.

Considering the new transformation $g(t,z) = j^{\delta}(t,z)$, we deduce the following linear parabolic differential equation:

$$j_t + \frac{\beta_t^2}{2} j_{zz} + \left[\alpha_t - \frac{\rho\beta_t(\mu_t - r_t)}{\sigma_t}\right] j_z - \frac{(1-\rho^2)}{2} \frac{(\mu_t - r_t)^2}{\sigma_t^2} j = 0 \quad (10)$$

for the value of the parameter δ satisfying:

$$\delta = \frac{1}{1-\rho^2} \quad (11)$$

we will call δ the distortion power, with reference to Zariphopoulou [2].

***Proposition* 1**
We assume that $(t,z) \mapsto \alpha_t, (t,z) \mapsto \beta_t, (t,z) \mapsto -\frac{\rho\beta_t(\mu_t - r_t)}{\sigma_t}$ and $(t,z) \mapsto -\frac{1-\rho^2}{2}\frac{(\mu_t - r_t)^2}{\sigma_t^2}$ are bounded and uniformly Hölder's continuous in $[0,T] \times R$. Moreover, we assume that β_t is uniformly elliptic. Then the value function Ψ is twice continuously differentiable with respect to $(\lambda, z) \in R_+ \times R$ and continuously differentiable with respect to t for $t \in [0,T)$.

***Proposition* 2**
i. The value function is given by:

$$\Psi(t,\lambda,z) = -e^{-\gamma\lambda e^{\int_t^T r_s ds}} j(t,z)^{\frac{1}{1-\rho^2}} \quad (12)$$

with

$$j(t,z)=E_Q\left(e^{-\frac{1-\rho^2}{2}\int_t^T\frac{(\mu_s-r_s)^2}{\sigma_s^2}ds}\middle|Z_t=z\right) \quad (13)$$

and Q the risk-neutral measure, such that

$dZ_s=\left(\alpha_s-\frac{\rho\beta_s(\mu_s-r_s)}{\sigma_s}\right)ds+\beta_s d\widetilde{W_s}$

where $\widetilde{W_s}=W_s+\int_0^s\frac{\rho\beta_u(\mu_u-r_u)}{\sigma_u}du$ is a Brownian motion under risk-neutral measure Q.

ii. The optimal control $\pi_t^\star=\pi^\star(t,z)$ is given by

$$\pi_t^\star=\frac{1}{\gamma}e^{-\int_t^T r_s ds}\left(\frac{\mu_t-r_t}{\sigma_t^2}+\frac{\rho}{1-\rho^2}\frac{\beta_t}{\sigma_t}\frac{j_z}{j}\right). \quad (14)$$

***Remark* 1**

The value function Ψ is concave and non-decreasing with respect to the wealth variable λ due to its close dependence on the utility function u.

Remark 2

The value function Ψ is a constrained viscosity solution of (46) with terminal condition $\Psi(T,\lambda,z)=-e^{-\gamma\lambda}$.

For the next, we assume that uniformly in $z\in R$ and $t\in[0,T]$, the volatility coefficient σ_t satisfies $\sigma_t\geq c_1$ for some constant $c_1>0$, and for some positive constant c_2 we have:

$$\frac{(\mu_t-r_t)^2}{\sigma_t^2}\leq c_2 \quad (15)$$

and that this condition (15) will be used to determine the growth conditions for the value function and to facilitate relevant verification results.

The following theorem provides a verification result for the value function.

***Theorem* 1**

The value function Ψ is given by $\Psi(t,\lambda,z)=-e^{-\gamma\lambda e^{\int_t^T r_s ds}}j^\delta(t,z)$ where j is the unique viscosity solution of (10) with terminal condition $j(T,z)=1$ and δ is given in (11).

Proof. When we apply the results of [7], It is easy to conclude that equation [10] satisfying the boundary and terminal conditions $j(T,z)=1$, has a unique viscosity solution. In fact, considering condition [15], the function j verifies $j\leq e^{\frac{(1-\rho^2)c_2}{2}(T-t)}$ and by applying the definition of viscosity solutions, we obtain directly that $\Gamma(t,\lambda,z)=-e^{-\gamma\lambda e^{\int_t^T r_s ds}}j^\delta(t,z)$ is a viscosity solution of the equation HJB (46) in $[0,T]\times R_+\times R$. The sub-solution property of viscosity is automatically satisfied at the limit point $\lambda=0$, where the slope of Γ is finite. Thus, Γ is a viscosity constrained solution of the HJB equation (46), belonging to the appropriate class of solutions where uniqueness has been established. We obtain that Γ coincides with the value function, and therefore Ψ is effectively given by the proposed closed solution.

III. Corporate Bond price and CDS spread by utility indifference with stochastic drift and volatility

During the investment period, the agent can invest part of his wealth in a corporate bond or a CDS and the rest in the portfolio of non-defaulting risky assets A_t^1 and non-risky asset M_t with dynamics defined above. The default on the reference entity is modeled by a Poisson process N_t with intensity κ (since our credit risk default model is of reduced form [12]) and the default time denoted τ_d is defined by: $\tau_d=inf\{t\geq 0; N_t=1\}$.

Let us construct this section by evaluating by the utility indifference method the price of the corporate bond $\mathcal{C}_t$ for a portfolio with non-risky assets and risky assets without default. Let us specify that this price is the one that provides the investor with the same level of expected utility when he invests the rest of his wealth $\lambda-\mathcal{C}_t$ in non-risky assets M_t and risky assets of values A_t^1 or when he invests all his wealth λ in these same assets. By buying a corporate bond, the investor receives a notional amount of F at maturity if the reference entity does not default before the maturity T; or receives a percentage R (assumed here as a random variable independent on $(0,1)$ of Brownian motion B_t) of the notional amount in the event of default before maturity.

Since for future cash flows, the certainty equivalent (see [4] and [6]) is the amount we would be willing to receive without risk, relative to expected future cash flows. The net present value of an investment can then be defined as the sum of certain cash flow equivalents discounted at the risk-free rate. The certainty equivalent of R verifies $E\left[u\left(RFe^{\int_t^T r_s ds}\right)\right]=u\left(\widetilde{R_t}Fe^{\int_t^T r_s ds}\right)$ in [4,39] and is defined by

$$\widetilde{R_t}=-\frac{1}{\gamma Fe^{\int_t^T r_s ds}}\ln E\left[e^{-\gamma RFe^{\int_t^T r_s ds}}\right].$$

The investor's wealth dynamic with contingent claim is given by:

$$\begin{cases} d\overline{\Lambda}_s=\left[r_s\overline{\Lambda}_s+\overline{\pi}_s(\mu_s-r_s)\right]ds+\overline{\pi}_s\sigma_s dB_s \\ \overline{\Lambda}_\tau=\overline{\Lambda}_{\tau^-}+RF.\mathbf{1}_{\{\tau\leq T\}}+F.\mathbf{1}_{\{\tau>T\}} \end{cases} \quad (16)$$

where $\tau=\min(\tau_d,T)$ and limit wealth represents the fact that if the default on the reference entity occurs before maturity $(\tau\leq T)$ then the investor receives a random percentage R of the notional and in the opposite case $(\tau>T)$ he receives the whole notional.

The value function of the investor for a portfolio with contingent claim is

$$\overline{\Psi}(t,\lambda,z)=\sup_{\overline{\pi}_t\in\mathcal{A}}E\left[u(\overline{\Lambda}_T)\middle|\overline{\Lambda}_t=\lambda,Z_t=z,t<\tau_d\right]. \quad (17)$$

The value function (17) associated with the state process (16) satisfies the following HJB equation

$$\frac{\partial \overline{\Psi}}{\partial t}(t,\lambda,z) + \sup_{\overline{\pi}_t \in R} \mathcal{G}^{\overline{\pi}} \overline{\Psi}(t,\lambda,z) = 0, \quad (18)$$

with $\overline{\Psi}(T,\lambda,z) = u(\lambda + F), \forall \lambda \in R$ and where

$$\begin{aligned}\mathcal{G}^{\overline{\pi}}\overline{\Psi}(t,\lambda,z) = {}& [r_t\lambda + \overline{\pi}_t(\mu_t - r_t)]\frac{\partial \overline{\Psi}}{\partial \lambda}(t,\lambda,z) \\ &+ \frac{\sigma_t^2 \overline{\pi}_t}{2}\frac{\partial^2 \overline{\Psi}}{\partial \lambda^2} + \alpha_t \frac{\partial \overline{\Psi}}{\partial z}(t,\lambda,z) \\ &+ \frac{\beta_t^2}{2}\frac{\partial^2 \overline{\Psi}}{\partial z^2}(t,\lambda,z) \\ &+ \rho\beta_t\overline{\pi}_t\sigma_t \frac{\partial^2 \overline{\Psi}}{\partial z\,\partial \lambda}(t,\lambda,z) \\ &+ \kappa\left[\Psi\left(t,\lambda + \widetilde{R_t}F, z\right) - \overline{\Psi}(t,\lambda,z)\right].\end{aligned}$$

Similarly, to Section **II**, the value function is represented by $\overline{\Psi}(t,\lambda,z) = -e^{-\gamma\lambda e^{\int_t^T r_s ds}} l(t,z)^{\omega}$ where $l \in C^{1,2}([0,T]\times R)$ and verifies the nonlinear parabolic equation

$$l_t + \frac{\beta_t^2}{2} l_{zz} + \left[\alpha_t - \frac{\rho\beta_t(\mu_t - r_t)}{\sigma_t}\right] l_z + \left[\frac{(\mu_t - r_t)^2}{2\omega\sigma_t^2} + \frac{\kappa}{\omega}\right] l + \frac{\beta_t^2}{2}(-\rho^2\omega + \omega - 1)\frac{l_z^2}{l} + \frac{\kappa}{\omega} e^{-\gamma\widetilde{R_t}F e^{\int_t^T r_s ds}} \frac{g}{l^{\omega-1}} = 0, \quad (19)$$

with terminal condition $l(T,z) = e^{-\frac{\gamma F}{\omega}}$, such that the proof of (19) can be found in the APPENDIX in *Proof 2* in the subsection *Some Proofs*, with g the function defined in section **II** by $g(t,z) = j(t,z)^{\frac{1}{1-\rho^2}}$ and j defined by (13). For the value of the parameter (another distortion power) ω satisfying:

$$\omega = \frac{1}{1-\rho^2} \quad (20)$$

this equation (18) becomes the following semi linear parabolic differential equation

$$l_t + \frac{\beta_t^2}{2} l_{zz} + \left[\alpha_t - \frac{\rho\beta_t(\mu_t - r_t)}{\sigma_t}\right] l_z - (1-\rho^2)\left[\frac{(\mu_t - r_t)^2}{2\sigma_t^2} + \kappa\right] l + \kappa(1-\rho^2) e^{-\gamma\widetilde{R_t}F e^{\int_t^T r_s ds}} j^{\frac{1}{1-\rho^2}} l^{-\frac{\rho^2}{1-\rho^2}} = 0, \quad (21)$$

with terminal condition $l(T,z) = e^{-\gamma(1-\rho^2)F}$.

***Remark*3**

$\forall \rho \in (-1,1), -\frac{\rho^2}{1-\rho^2} \le 0$. In particular, if $\rho = 0$ then $-\frac{\rho^2}{1-\rho^2} = 0, \frac{1}{1-\rho^2} = 1$ and (21) becomes

$$l_t + \frac{\beta_t^2}{2} l_{zz} + \alpha_t l_z - \left[\frac{(\mu_t - r_t)^2}{2\sigma_t^2} + \kappa\right] l + \kappa e^{-\gamma\widetilde{R_t}F e^{\int_t^T r_s ds}} j = 0 \quad (22)$$

with $l(T,z) = e^{-\gamma F}$.

***Proposition* 3**

iii. The solution of (22) is given by:

$$l(t,z) = e^{\kappa t} E\left[e^{-(\gamma F + \kappa T)} e^{-\int_t^T \frac{(\mu_s - r_s)^2}{2\sigma_s^2} ds} + \kappa \int_t^T e^{-\Gamma_s} e^{-\int_t^s \frac{(\mu_u - r_u)^2}{2\sigma_u^2} du} j(s,z) ds \middle| Z_t = z\right] \quad (23)$$

with $\forall t \in [0,T], \Gamma_t = \kappa t + \gamma\widetilde{R_t}F e^{\int_t^T r_s ds}$ and $(t,z) \mapsto j(t,z)$ given by

$$j(t,z) = E\left[e^{-\frac{1}{2}\int_t^T \frac{(\mu_s - r_s)^2}{\sigma_s^2} ds} \middle| Z_t = z\right]. \quad (24)$$

iv. The optimal control $\overline{\pi}_t^\star = \overline{\pi}^\star(t,z)$ in this particular case is

$$\overline{\pi}_t^\star = \frac{1}{\gamma} e^{-\int_t^T r_s ds} \frac{(\mu_t - r_t)}{\sigma_t^2}. \quad (25)$$

In the following theorem, we can define the price of the corporate bond when $\rho = 0$.

***Theorem* 2**

The utility indifference price $\mathcal{C}_t = \mathcal{C}(t,z)$ of corporate bond is given by

$$\mathcal{C}_t = \frac{1}{\gamma} e^{-\int_t^T r_s ds} \ln\left(\frac{j(t,z)}{l(t,z)}\right) \quad (26)$$

Proof. We know that an indifference price P is the price for which an agent (an investor in our case) would have the same level of expected utility when he invests the rest of his wealth $\lambda - P$ in his portfolio as by not doing so.

We can write

$$\begin{aligned}\overline{\Psi}(t,\lambda - P,z) = \Psi(t,\lambda,z) &\Leftrightarrow -e^{-\gamma(\lambda - P)e^{\int_t^T r_s ds}} l(t,z) \\ &= -e^{-\gamma\lambda e^{\int_t^T r_s ds}} j(t,z) \\ &\Leftrightarrow e^{\gamma P e^{\int_t^T r_s ds}} l(t,z) = j(t,z) \\ &\Leftrightarrow \gamma P e^{\int_t^T r_s ds} = \ln\left(\frac{j(t,z)}{l(t,z)}\right) \\ &\Leftrightarrow P = \frac{1}{\gamma} e^{-\int_t^T r_s ds} \ln\left(\frac{j(t,z)}{l(t,z)}\right)\end{aligned}$$

Thus, considering $P = \mathcal{C}_t$, we obtain the result.

This following section allows us to evaluate continuous CDS premium that the investor pays to the protection seller.

In the following, we now assume that the investor buys a CDS and pays a continuous premium rate $S_p(t)$ paid on the notional amount F from the time the contract was established until maturity or the time of default of the reference entity, whichever comes first. If default occurs before maturity, the investor receives a random payment of $(1-R)F$ and all future premium payments cease.

Similar to the equation (16), we have the dynamics of the investor's wealth with the CDS premium rate given by the equations (27) and (28) and define by:

$$d\widetilde{\Lambda_t} = \left[r_t\widetilde{\Lambda_t} + \epsilon S_p(t)F + \widetilde{\pi_t}(\mu_t - r_t)\right]dt + \widetilde{\pi_t}\sigma_t dB_t, \quad 0 < t < \tau \quad (27)$$

$$d\widetilde{\Lambda_t} = \left[r_t\widetilde{\Lambda_t} + \widetilde{\pi_t}(\mu_t - r_t)\right]dt + \widetilde{\pi_t}\sigma_t dB_t, \ t > \tau \quad (28)$$

where the limit wealth $\widetilde{\Lambda_\tau} = \widetilde{\Lambda_{\tau^-}} - \epsilon(1-R)F.1_{\{\tau_d \le T\}}$.

Here, $\epsilon = +1$ for the CDS seller and $\epsilon = -1$ for the buyer.

The value function of the investor for a portfolio with the CDS premium rate is

$$\widetilde{\Psi}(t,\lambda,z) = \sup_{\widetilde{\pi_t}\in\mathcal{A}} E\left[u\left(\widetilde{\Lambda_T}\right)\middle|\widetilde{\Lambda_t} = \lambda, Z_t = z, t < \tau_d\right]. \quad (29)$$

The value function (29) associated with the states (27) and (28) satisfies the following HJB equation

$$\frac{\partial\widetilde{\Psi}}{\partial t}(t,\lambda,z) + \sup_{\widetilde{\pi_t}\in R} \mathcal{G}_\epsilon^{\widetilde{\pi}}\, \widetilde{\Psi}(t,\lambda,z) = 0, \quad (30)$$

with $\widetilde{\Psi}(T,\lambda,z) = u(\lambda),\ \lambda \in R$ and where

$$\begin{aligned}\mathcal{G}_\epsilon^{\widetilde{\pi}}\widetilde{\Psi}(t,\lambda,z) = \Big[&r_t\lambda + \widetilde{\pi_t}(\mu_t - r_t) + \epsilon\, S_p(t)F\Big]\frac{\partial\widetilde{\Psi}}{\partial\lambda}(t,\lambda,z) \\ &+ \frac{\widetilde{\pi_t}^2\sigma_t^2}{2}\frac{\partial^2\widetilde{\Psi}}{\partial\lambda^2}(t,\lambda,z) + \alpha_t\frac{\partial\widetilde{\Psi}}{\partial z}(t,\lambda,z) \\ &+ \frac{\beta_t^2}{2}\frac{\partial^2\widetilde{\Psi}}{\partial z^2}(t,\lambda,z) \\ &+ \rho\beta_t\widetilde{\pi_t}\sigma_t\frac{\partial^2\widetilde{\Psi}}{\partial z\partial\lambda}(t,\lambda,z) \\ &+ \kappa\Big[\Psi\big(t,\lambda - \epsilon\big(1-\widetilde{R_t}\big)F, z\big) \\ &- \widetilde{\Psi}(t,\lambda,z)\Big]\end{aligned}$$

as before, we consider $\widetilde{\Psi}$ in the form

$$\widetilde{\Psi}(t,\lambda,z) = -e^{-\gamma\lambda e^{\int_t^T r_s ds}}\tilde{l}(t,z)^{\theta}$$

where $\tilde{l} \in C^{1,2}([0,T]\times R)$ and verifies the nonlinear parabolic equation

$$\widetilde{l_t} + \frac{\beta_t^2}{2}\tilde{l}_{zz} + \left[\alpha_t - \frac{\rho\beta_t(\mu_t - r_t)}{\sigma_t}\right]\widetilde{l_z} - \left[\frac{(\mu_t - r_t)^2}{2\theta\sigma_t^2} + \frac{\kappa}{\theta} + \epsilon\frac{\gamma}{\theta}S_p(t)Fe^{\int_t^T r_s ds}\right]\tilde{l} + \frac{\beta_t^2}{2}(-\rho^2\theta + \theta - 1)\frac{\tilde{l}_z^2}{\tilde{l}} + \frac{\kappa}{\theta}e^{\epsilon\gamma(1-\widetilde{R_t})Fe^{\int_t^T r_s ds}}\frac{g}{\tilde{l}^{\theta-1}} = 0 \quad (31)$$

For the value of the parameter (another distortion power) θ satisfying:

$$\theta = \frac{1}{1-\rho^2}$$

this equation (31) becomes the following semi linear parabolic differential equation

$$\widetilde{l_t} + \frac{\beta_t^2}{2}\tilde{l}_{zz} + \left[\alpha_t - \frac{\rho\beta_t(\mu_t - r_t)}{\sigma_t}\right]\widetilde{l_z} - (1-\rho^2)\left[\frac{(\mu_t - r_t)^2}{2\sigma_t^2} + \kappa + \epsilon\gamma S_p(t)Fe^{\int_t^T r_s ds}\right]\tilde{l} + (1-\rho^2)\kappa e^{\epsilon\gamma(1-\widetilde{R_t})Fe^{\int_t^T r_s ds}} j^{\frac{1}{1-\rho^2}}\tilde{l}^{-\frac{\rho^2}{1-\rho^2}} = 0, \quad (32)$$

with terminal condition $\tilde{l}(T,z) = 1$ and for the same particular reasons as in remark (***Remark* 2**), equation (32) becomes:

$$\widetilde{l_t} + \frac{\beta_t^2}{2}\tilde{l}_{zz} + \alpha_t\widetilde{l_z} - \left[\frac{(\mu_t - r_t)^2}{2\sigma_t^2} + \kappa + \epsilon\gamma S_p(t)Fe^{\int_t^T r_s ds}\right]\tilde{l} + \kappa e^{\epsilon\gamma(1-\widetilde{R_t})Fe^{\int_t^T r_s ds}} j = 0 \quad (33)$$

with $\tilde{l}(T,z) = 1$.

***Proposition* 4**

v. The solution of (33) is given by:

$$\tilde{l}(t,z) = E\left[e^{-\int_t^T V(s,z)ds} + \kappa\int_t^T e^{\epsilon\gamma(1-\widetilde{R_s})Fe^{\int_s^T r_u du}} j(s,z)e^{-\int_t^s V(u,z)du}\, ds \,\middle|\, Z_t = z\right] \quad (34)$$

with $V(t,z) = \frac{(\mu_t - r_t)^2}{2\sigma_t^2} + \kappa + \epsilon\gamma S_p(t)Fe^{\int_t^T r_s ds}$

vi. The optimal control $\widetilde{\pi_t}^\star = \widetilde{\pi}^\star(t,z)$ in this particular case is

$$\widetilde{\pi_t}^\star = \frac{1}{\gamma}e^{-\int_t^T r_s ds}\frac{(\mu_t - r_t)}{\sigma_t^2}. \quad (35)$$

***Theorem* 3**

The utility indifference spread $S_p(t)$ of CDS is implicitly given by the following $P - a.s$ equation

$$\kappa e^{\kappa t + \int_t^T \frac{(\mu_s - r_s)^2}{2\sigma_s^2}ds}\int_t^T j(s,z)\, e^{-\Delta_s - \int_t^s\left(\frac{(\mu_u - r_u)^2}{2\sigma_u^2} - \epsilon\gamma F S_p(u)e^{\int_u^T r_v dv}\right)du} ds + e^{-\kappa(T-t) - \gamma F\int_t^T S_p(s)e^{\int_s^T r_u du}ds} = 1 \quad (36)$$

where for $t \in [0,T]; \Delta_t = -\kappa t - \gamma\big(1 - \widetilde{R_t}\big)Fe^{\int_t^T r_s ds}$.

Proof. Similar to the price of the defaultable bond, the indifference credit default swap spread is defined as the value $S_p(t)$ satisfying the equation $\widetilde{\Psi}(t,\lambda,z) = \Psi(t,\lambda,z) \Leftrightarrow \tilde{l}(t,z) = j(t,z)$, when the coefficient correlation $\rho = 0$ and leading to the desired equation.

***Remark* 3**

Under the power utility function, Zariphopoulou Thaleia in [2] obtains a distortion power δ that depends on the risk aversion coefficient γ and the correlation ρ between the brownian motions that modulate the stock price A_t^1 and the factor process Z_t respectively. On the other hand, in our work and that of Boguslavskaya Elena and Muravey Dmitry in [8], under the exponential utility function, we obtain a distortion power depending solely on the correlation coefficient.

IV. Simulation Curves of the Paper

A. *Distortion Powers as a Function of Correlation*

All the distortion powers δ, ω and θ in our problem are equal and positive and defined by: $\delta = \omega = \theta = \frac{1}{1-\rho^2}$ on the interval $[-1;1]$. A representation of these distortion powers is given in the following figure:

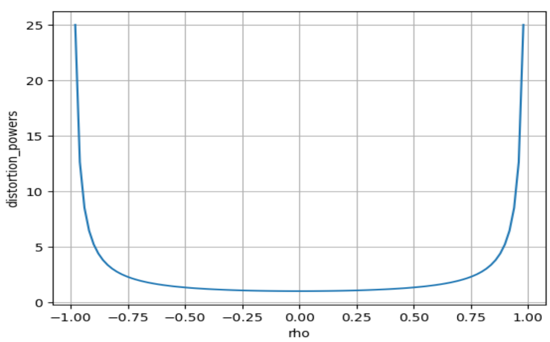

Fig.1. Distortion powers as a function of correlation.

B. Certainty Equivalent $\left(\widetilde{R_t}\right)_{t\in[0,T]}$ of Random Rate R in the Event of Default and Instantaneous Variance $(Z_t)_{t\in[0,T]}$.

We present below the representative trajectories of the certainty equivalent of the rate R and of the instantaneous variance process which governs the volatility of the factor process (risky asset). For all R_t trajectories, the risk aversion coefficient $\gamma = 0.7$.

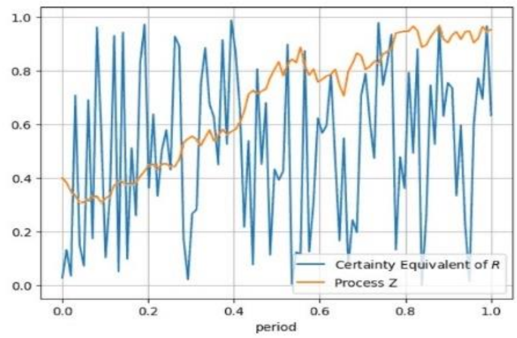

Fig.2.α= $0.7, \beta = 0.5, F = 500, T = 1 for$
$dZ_t = \alpha dt + \beta dZ_t, Z_0 = 0.4$.

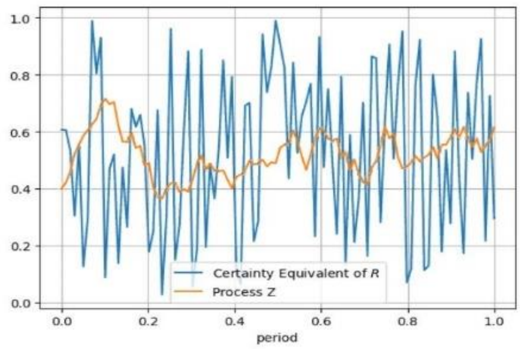

Fig.3.$\alpha = 0.7, \beta = 0.5, F = 500, T = 1$ *for*

$\mathrm{d}Z_t = \alpha(\beta - Z_t)dt + \beta\sqrt{Z_t}dW_t,\ Z_0 = 0.4.$

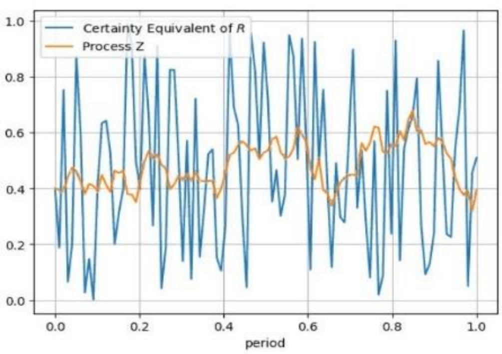

Fig.4.$\alpha = 0.7, \beta = 0.5, F = 500, T = 1$ *for*

$\mathrm{d}Z_t = \alpha(\beta - Z_t)dt + \beta dW_t,\ Z_0 = 0.4.$

C. Utility Indifference Price $\mathcal{C}_t$ of Contingent Claim.

We present above in Fig.5 and Fig.6 the utility indifference price surface of the contingent claim $\mathcal{C}_t$ as conditional expectation knowing the variation of the factor stochastic process Z_t.

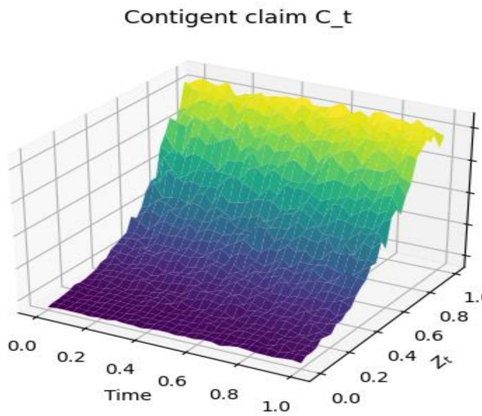

Fig.5.$\alpha = 0.7, \beta = 0.5, F = 500, T = 1$ *for*
$\mathrm{d}Z_t = \alpha(\beta - Z_t)dt + \beta dW_t,\ Z_0 = 0.4$

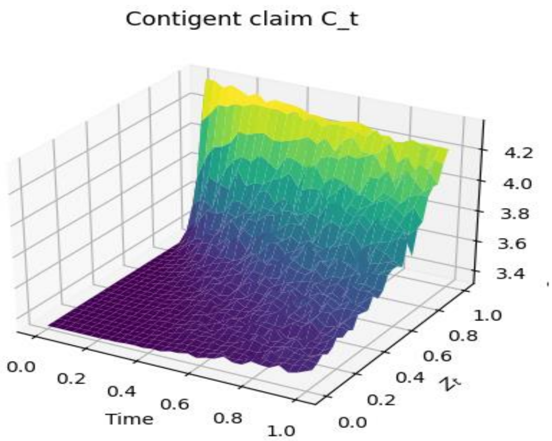

Fig.6.$\alpha = 0.7, \beta = 0.5, F = 500, T = 1$ *for*
$\mathrm{d}Z_t = \alpha(\beta - Z_t)dt + \beta\sqrt{Z_t}dW_t,\ Z_0 = 0.4.$

Unlike the numerical analysis of bond yield term structures with several levels of risk aversion presented by Sigloch [4], in our case, we set the coefficient of risk aversion to $\gamma = 0.7$ and modify the dynamics of the underlying process A_t and the factor process Z_t, and then observe the surface $\mathcal{C}_t$.

APPENDIX

A. Definition of Viscosity Solutions of Non-Linear Parabolic Partial Differential Equations.

We note that the notion of viscosity solutions was introduced by [9] for first-order equations and by [10] for second-order equations. However, a general overview of the theory can be found in [11].

Consider a non-linear second-order partial differential equation of the form

$$G(X, v, Dv, D^2v) = 0 \text{ in } [0,T] \times \Omega \quad (37)$$

in which $\Omega \subseteq R^2, Dv$ and D^2v denote the gradient vector and the second-derivative matrix of v, and the function G is continuous in all its arguments and degenerate elliptic, meaning that

$$G(X, p, q, A + B) \leq G(X, p, q, A) \text{ if } B \geq 0$$

A continuous function $v: [0,T] \times \overline{\Omega} \to R$ is a viscosity solution of (37) if the following two conditions hold:

a) v is a viscosity subsolution of (37) on $[0,T] \times \overline{\Omega}$; that is, if for any $\phi \in C^{1,2}([0,T] \times \overline{\Omega})$ and any local maximum point $X_0 \in [0,T] \times \overline{\Omega}$ of $v - \phi$, $G(X_0, v(X_0), D\phi(X_0), D^2\phi(X_0)) \leq 0$.

b) v is a viscosity supersolution of (37) on $[0,T] \times \overline{\Omega}$; that is, if for any $\phi \in C^{1,2}([0,T] \times \Omega)$ and any local minimum point $X_0 \in [0,T] \times \overline{\Omega}$ of $v - \phi$, $G(X_0, v(X_0), D\phi(X_0), D^2\phi(X_0)) \geq 0$.

B. *Some Proofs.*

Proof.1. First, we construct the partial differential equation verified by Ψ.

Let's move on to double inequality:

$\Longrightarrow$) Show first that,

$$\frac{\partial \Psi}{\partial t}(t, X_t^\pi) + \sup_{\pi_t \in \mathcal{A}} \mathcal{G}^\pi \Psi(t, X_t^\pi) \leq 0.$$

Let $n \in N^\star$, the process $X_t^\pi = \left(X_t^{\pi,i}\right)_{(i,t)\in\{1,2\}\times[0,T]}$, and the function $\pi = \pi_t \in \mathcal{A}$ (where $\mathcal{A}$ is the set of admissible policies of this problem) such that

$$dX_t^{\pi,i} = u^i(t, X_t^\pi, \pi_t)dt + \sum_{i'=1}^{n} v^{ii'}(t, X_t^\pi, \pi_t) dB_t^{i'}$$

and by using the Ito's multidimensional formula on a function Ψ,

$$d\Psi(t, X_t^\pi) = \left[\frac{\partial \Psi}{\partial t}(t, X_t^\pi) + \sum_{i=1}^{n} \frac{\partial \Psi}{\partial x_i}(t, X_t^\pi) u^i(t, X_t^\pi, \pi_t)\right] dt + \frac{1}{2}\sum_{i,i'=1}^{n} \frac{\partial^2 \Psi}{\partial x_i \partial x_{i'}}(t, X_t^\pi) \sum_{k=1}^{n} v^{ik} v^{ki'}(t, X_t^\pi, \pi_t) corr\left(B_t^i; B_t^{i'}\right) dt + \sum_{i=1}^{n} \frac{\partial \Psi}{\partial x_i}(t, X_t^\pi) \sum_{i'=1}^{n} v^{ii'}(t, X_t^\pi, \pi_t) dB_t^{i'}.$$

then by integrating between t and $t + h$, we obtain:

$$\Psi(t+h, X_{t+h}^\pi) = \Psi(t, X_t^\pi) + \int_t^{t+h} \left[\frac{\partial \Psi}{\partial t}(s, X_s^\pi) + \sum_{i=1}^{n} \frac{\partial \Psi}{\partial x_i}(s, X_s^\pi) u^i(s, X_s^\pi, \pi_s) + \frac{1}{2}\sum_{i,i'=1}^{n} \frac{\partial^2 \Psi}{\partial x_i \partial x_{i'}}(s, X_s^\pi) \sum_{k=1}^{n} v^{ik} v^{ki'}(s, X_s^\pi, \pi_s) corr\left(B_s^i; B_s^{i'}\right)\right] ds + \int_t^{t+h} \sum_{i=1}^{n} \frac{\partial \Psi}{\partial x_i}(s, X_s^\pi) \sum_{i'=1}^{n} v^{ii'}(s, X_s^\pi, \pi_s) dB_s^{i'}.$$

We set the following expression, which represents the infinitesimal generator:

$$\mathcal{G}^\pi \Psi(s, X_s^\pi) = \sum_{i=1}^{n} \frac{\partial \Psi}{\partial x_i}(s, X_s^\pi) u^i(s, X_s^\pi, \pi_s) + \frac{1}{2}\sum_{i,i'=1}^{n} \frac{\partial^2 \Psi}{\partial x_i \partial x_{i'}}(s, X_s^\pi) \sum_{k=1}^{n} v^{ik} v^{ki'}(s, X_s^\pi, \pi_s) corr\left(B_s^i; B_s^{i'}\right)$$

and we then obtain the new Ito process:

$$\Psi(t+h, X_{t+h}^\pi) = \Psi(t, X_t^\pi) + \int_t^{t+h} \left[\frac{\partial \Psi}{\partial t}(s, X_s^\pi) + \mathcal{G}^\pi \Psi(s, X_s^\pi)\right] ds + \int_t^{t+h} \sum_{i=1}^{n} \frac{\partial \Psi}{\partial x_i}(s, X_s^\pi) \sum_{i'=1}^{n} v^{ii'}(s, X_s^\pi, \pi_s) dB_s^{i'}.$$

Thus, applying conditional expectation on the part and other sides of this equality with $x = (x_1, x_2, \cdots, x_n)$, we obtain:

$$E[\Psi(t+h, X_{t+h}^\pi) | X_t^\pi = x] = E\left[\Psi(t, X_t^\pi) + \int_t^{t+h} \left[\frac{\partial \Psi}{\partial t}(s, X_s^\pi) + \mathcal{G}^\pi \Psi(s, X_s^\pi)\right] ds \,\middle|\, X_t^\pi = x\right] + E\left[\int_t^{t+h} \sum_{i=1}^{n} \frac{\partial \Psi}{\partial x_i}(s, X_s^\pi) \sum_{i'=1}^{n} v^{ii'}(s, X_s^\pi, \pi_s) dB_s^{i'} \,\middle|\, X_t^\pi = x\right]$$

since the expectation of the Ito's stochastic integral is zero, the result is

$$E\left[\int_t^{t+h} \sum_{i=1}^{n} \frac{\partial \Psi}{\partial x_i}(s, X_s^\pi) \sum_{i'=1}^{n} v^{ii'}(s, X_s^\pi, \pi_s) dB_s^{i'} \,\middle|\, X_t^\pi = x\right] = 0 \quad (38)$$

We will get

$$E[\Psi(t+h, X_{t+h}^\pi) | X_t^\pi = x] = \Psi(t, x) + E\left[\int_t^{t+h} \left[\frac{\partial \Psi}{\partial t}(s, X_s^\pi) + \mathcal{G}^\pi \Psi(s, X_s^\pi)\right] ds \,\middle|\, X_t^\pi = x\right] \quad (39)$$

because the expectation is linear and

$E[\Psi(t, X_t^{\pi}) | X_t^{\pi} = x] = \Psi(t, x)$.

By definition of the value function,

$$\Psi(t, x) \geq E[\Psi(t + h, X_{t+h}^{\pi}) | X_t^{\pi} = x] \quad (40)$$

Thus from (39) and (40) we get

$$E\left[\int_t^{t+h} \left[\frac{\partial \Psi}{\partial t}(s, X_s^{\pi}) + \mathcal{G}^{\pi}\Psi(s, X_s^{\pi})\right] ds \,\middle|\, X_t^{\pi} = x\right] \leq 0 \quad (41)$$

From Equation (41), we deduce an inequality almost surely under the probability measure P. By dividing it by h and letting h approach 0, we obtain this expression:

$$\left[\frac{\partial \Psi}{\partial t}(t, X_t^{\pi}) + \mathcal{G}^{\pi}\Psi(t, X_t^{\pi})\right] \leq 0 \quad (42)$$

this is true for all $\pi = \pi_t \in \mathcal{A}$, and we will get:

$$\frac{\partial \Psi}{\partial t}(t, X_t^{\pi}) + \sup_{\pi_t \in \mathcal{A}} \mathcal{G}^{\pi}\Psi(t, X_t^{\pi}) \leq 0$$
(43)

We still need to prove the second inequality.

$\Leftarrow$) Then prove that,

$$\left[\frac{\partial \Psi}{\partial t}(t, X_t^{\pi}) + sup_{\pi_t \in \mathcal{A}} \mathcal{G}^{\pi}\Psi(t, X_t^{\pi})\right] \geq 0$$

We can consider that $\pi^{\star} = \pi_t^{\star}$ is the optimal control:

$$\Psi(t, x) = E\left[\Psi\left(t + h, X_{t+h}^{\pi^{\star}}\right) \middle| X_t^{\pi^{\star}} = x\right]$$

where $X^{\pi^{\star}}$, and applying the Ito formula between t and $t + h$, we get:

$$\Psi\left(t + h, X_{t+h}^{\pi^{\star}}\right) = \Psi\left(t, X_t^{\pi^{\star}}\right)$$

$$+ \int_t^{t+h} [\frac{\partial \Psi}{\partial t}\left(s, X_s^{\pi^{\star}}\right) + \sum_{i=1}^{n} \frac{\partial \Psi}{\partial x_i}\left(s, X_s^{\pi^{\star}}\right) u^i\left(s, X_s^{\pi^{\star}}, \pi_s^{\star}\right) + \frac{1}{2}\sum_{i,i'=1}^{n} \frac{\partial^2 \Psi}{\partial x_i \partial x_{i'}}\left(s, X_s^{\pi^{\star}}\right) \sum_{k=1}^{n} v^{ik}\, v^{ki'}\left(s, X_s^{\pi^{\star}}, \pi_s^{\star}\right) corr\left(B_s^i; B_s^{i'}\right)] ds + \int_t^{t+h} \sum_{i=1}^{n} \frac{\partial \Psi}{\partial x_i}\left(s, X_s^{\pi^{\star}}\right) \sum_{i'=1}^{n} v^{ii'}\left(s, X_s^{\pi^{\star}}, \pi_s^{\star}\right) dB_s^{i'}.$$

Taking conditional expectation, also by using the definition of our value function, then by dividing it by h and letting h approach 0, the following expression:

$$E\left[\Psi\left(t + h, X_{t+h}^{\pi^{\star}}\right) \middle| X_t^{\pi^{\star}} = x\right] = \Psi(t, x) + E\left[\int_t^{t+h} \left[\frac{\partial \Psi}{\partial t}\left(s, X_s^{\pi^{\star}}\right) + \mathcal{G}^{\pi^{\star}}\Psi\left(s, X_s^{\pi^{\star}}\right)\right] ds \,\middle|\, X_t^{\pi^{\star}} = x\right]$$

allows us to write:

$$\left[\frac{\partial \Psi}{\partial t}\left(t, X_t^{\pi^{\star}}\right) + \mathcal{G}^{\pi^{\star}}\Psi\left(t, X_t^{\pi^{\star}}\right)\right] = 0$$

leading to:

$$\left[\frac{\partial \Psi}{\partial t}(t, X_t^{\pi}) + sup_{\pi_t \in \mathcal{A}} \mathcal{G}^{\pi}\Psi(t, X_t^{\pi})\right] \geq 0 \quad (44)$$

Using the (43) and (44) relationships, it can be concluded that:

$$\frac{\partial \Psi}{\partial t}(t, X_t^{\pi}) + \sup_{\pi_t \in \mathcal{A}} \mathcal{G}^{\pi}\Psi(t, X_t^{\pi}) = 0$$
(45)

After these two inequalities constructed, we make the following considerations for $n = 2$ and $i' \in \{1,2\}$:

$X_t^{\pi,1} = \Lambda_t$, $X_t^{\pi,2} = Z_t$, $B_t^1 = B_t$, $B_t^2 = W_t$, $u^1 = [r_t\Lambda_t + \pi_t(\mu_t - r_t)]$, $v^{11} = \sigma_t\pi_t$, $u^2 = \alpha_t$, $v^{22} = \beta_t$, $corr(B_t^1; B_t^2) = \rho$

in order to obtain the following PDE:

$$\begin{cases} \frac{\partial \Psi}{\partial t}(t, \lambda, z) + \sup_{\pi_t \in \mathcal{A}} \mathcal{G}^{\pi}\Psi(t, \lambda, z) = 0 \\ \Psi(T, \lambda, z) = u(\lambda), \quad \lambda \in R, \end{cases}$$
(46)

Where

$$\mathcal{G}^{\pi}\Psi(t, \lambda, z) = [r_t\lambda + \pi_t(\mu_t - r_t)]\frac{\partial \Psi}{\partial \lambda}(t, \lambda, z) + \frac{\sigma_t^2\pi_t}{2}\frac{\partial^2 \Psi}{\partial \lambda^2}(t, \lambda, z) + \alpha_t\frac{\partial \Psi}{\partial z}(t, \lambda, z) + \frac{\beta_t^2}{2}\frac{\partial^2 \Psi}{\partial z^2}(t, \lambda, z) + \rho\beta_t\pi_t\sigma_t\frac{\partial^2 \Psi}{\partial z \partial \lambda}(t, \lambda, z),$$

considering that $\Psi(t, \lambda, z) = -e^{-\gamma\lambda e^{\int_t^T r_s ds}} g(t, z)$, replacing the expression of Ψ in (46), we obtain:

$$-\gamma r_t\lambda e^{\int_t^T r_s ds} e^{-\gamma\lambda e^{\int_t^T r_s ds}} g - e^{-\gamma\lambda e^{\int_t^T r_s ds}} g_t + \sup_{\pi_t \in R}\{[r_t\lambda + \pi_t(\mu_t - r_t)]\gamma e^{\int_t^T r_s ds} e^{-\gamma\lambda e^{\int_t^T r_s ds}} g + \frac{\sigma_t^2\pi_t^2}{2}\left(-\gamma^2 e^{2\int_t^T r_s ds} e^{-\gamma\lambda e^{\int_t^T r_s ds}} g\right) + \alpha_t\left(-e^{-\gamma\lambda e^{\int_t^T r_s ds}} g_z\right) + \frac{\beta_t^2}{2}\left(-e^{-\gamma\lambda e^{\int_t^T r_s ds}} g_{zz}\right) + \rho\beta_t\sigma_t\pi_t\left(\gamma e^{\int_t^T r_s ds} e^{-\gamma\lambda e^{\int_t^T r_s ds}} g_z\right)\} = 0$$

This equation is equivalent to:

$$g_t + \sup_{\pi_t \in R}\{\left[-\pi_t(\mu_t - r_t)\gamma e^{\int_t^T r_s ds} + \frac{\sigma_t^2\pi_t^2}{2}\gamma^2 e^{2\int_t^T r_s ds}\right] g + \alpha_t g_z + \frac{\beta_t^2}{2} g_{zz} - \rho\beta_t\sigma_t\pi_t\gamma e^{\int_t^T r_s ds} g_z\} = 0 \quad (47)$$

By setting,

$$\eta(\pi_t) = \left[-\pi_t(\mu_t - r_t)\gamma e^{\int_t^T r_s ds} + \frac{\sigma_t^2\pi_t^2}{2}\gamma^2 e^{2\int_t^T r_s ds}\right] g + \alpha_t g_z + \frac{\beta_t^2}{2} g_{zz} - \rho\beta_t\sigma_t\pi_t\gamma e^{\int_t^T r_s ds} g_z$$

The first-order optimality condition:

$\frac{d\eta(\pi_t)}{d\pi_t} = 0 \Leftrightarrow$

$\left[-(\mu_t - r_t)\gamma e^{\int_t^T r_s ds} + \sigma_t^2 \pi_t \gamma^2 e^{2\int_t^T r_s ds}\right] g - \rho\beta_t\sigma_t\gamma e^{\int_t^T r_s ds} g_z = 0 \Leftrightarrow \sigma_t^2\pi_t\gamma e^{\int_t^T r_s ds} = (\mu_t - r_t) + \rho\beta_t\sigma_t \frac{g_z}{g}$

then the optimal control is

$$\pi_t^\star = \frac{1}{\gamma} e^{-\int_t^T r_s ds}\left(\frac{\mu_t - r_t}{\sigma_t^2} + \frac{\rho\beta_t}{\sigma_t}\frac{g_z}{g}\right). \quad (48)$$

Let introduce this expression in (47), we will get

$$g_t + \left[-\left(\frac{\mu_t - r_t}{\sigma_t^2} + \frac{\rho\beta_t}{\sigma_t}\frac{g_z}{g}\right)(\mu_t - r_t) + \frac{\sigma_t^2}{2}\left(\frac{\mu_t - r_t}{\sigma_t^2} + \frac{\rho\beta_t}{\sigma_t}\frac{g_z}{g}\right)^2\right] g + \alpha_t g_z + \frac{\beta_t^2}{2} g_{zz} - \rho\beta_t\sigma_t\left(\frac{\mu_t - r_t}{\sigma_t^2} + \frac{\rho\beta_t}{\sigma_t}\frac{g_z}{g}\right) g_z = 0$$

By a simple development, we can conclude that this is equivalent to:

$$g_t + \frac{\beta_t^2}{2} g_{zz} + \left[\alpha_t - \frac{\rho\beta_t(\mu_t r_t)}{\sigma_t}\right] g_z - \frac{1}{2}\left(\frac{\mu_t - r_t}{\sigma_t}\right)^2 - \frac{\rho^2\beta_t^2}{2}\frac{g_z^2}{g} = 0.$$

Proof.2. Given equation (18) of HJB

$$\frac{\partial\overline{\Psi}}{\partial t}(t,\lambda,z) + \sup_{\overline{\pi}_t \in R} \mathcal{G}^{\pi}\overline{\Psi}(t,\lambda,z) = 0, \quad (18)$$

with $\overline{\Psi}(T,\lambda,z) = u(\lambda + F), \forall\lambda \in R$ and where

$$\mathcal{G}^{\pi}\overline{\Psi}(t,\lambda,z) = [r_t\lambda + \overline{\pi}_t(\mu_t - r_t)]\frac{\partial\overline{\Psi}}{\partial\lambda}(t,\lambda,z) + \frac{\sigma_t^2\overline{\pi}_t}{2}\frac{\partial^2\overline{\Psi}}{\partial\lambda^2} + \alpha_t\frac{\partial\overline{\Psi}}{\partial z}(t,\lambda,z) + \frac{\beta_t^2}{2}\frac{\partial^2\overline{\Psi}}{\partial z^2}(t,\lambda,z) + \rho\beta_t\overline{\pi}_t\sigma_t\frac{\partial^2\overline{\Psi}}{\partial z\,\partial\lambda}(t,\lambda,z) + \kappa\left[\Psi(t,\lambda + \widetilde{R_t}F, z) - \overline{\Psi}(t,\lambda,z)\right],$$

the method of separating variables allows us to construct $\overline{\Psi}(t,\lambda,z) = -e^{-\gamma\lambda e^{\int_t^T r_s ds}} l(t,z)^{\omega}$ where $l \in C^{1,2}([0,T]\times R)$. Using this transformation of $\overline{\Psi}$, (18) becomes:

$-\gamma r_t\lambda e^{\int_t^T r_s ds} e^{-\gamma\lambda e^{\int_t^T r_s ds}} l^{\omega} - \omega e^{-\gamma\lambda e^{\int_t^T r_s ds}} l_t l^{\omega-1}$
$+\sup_{\overline{\pi}_t\in R}\{[r_t\lambda + \overline{\pi}_t(\mu_t - r_t)]\gamma e^{\int_t^T r_s ds} e^{-\gamma\lambda e^{\int_t^T r_s ds}} l^{\omega} + \frac{\sigma_t^2\overline{\pi}_t^2}{2}\left(-\gamma^2 e^{2\int_t^T r_s ds} e^{-\gamma\lambda e^{\int_t^T r_s ds}} l^{\omega}\right) + \alpha_t\left(-\omega e^{-\gamma\lambda e^{\int_t^T r_s ds}} l_z l^{\omega-1}\right) + \frac{\beta_t^2}{2}\left(-\omega e^{-\gamma\lambda e^{\int_t^T r_s ds}} l_{zz} l^{\omega-1} - \omega(\omega-1)e^{-\gamma\lambda e^{\int_t^T r_s ds}} l_z^2 l^{\omega-2}\right) + \rho\beta_t\sigma_t\overline{\pi}_t\left(\gamma\omega e^{\int_t^T r_s ds} e^{-\gamma\lambda e^{\int_t^T r_s ds}} l_z l^{\omega-1}\right) - \kappa e^{-\gamma(\lambda + \widetilde{R_t}F)e^{\int_t^T r_s ds}} g + \kappa e^{-\gamma\lambda e^{\int_t^T r_s ds}} l^{\omega}\} = 0$

equivalent to,

$$\omega l_t l^{\omega-1} + \sup_{\overline{\pi}_t\in R}\{-\overline{\pi}_t(\mu_t - r_t)\gamma e^{\int_t^T r_s ds} l^{\omega} + \frac{\sigma_t^2\overline{\pi}_t^2}{2}\left(\gamma^2 e^{2\int_t^T r_s ds} l^{\omega}\right) + \omega\alpha_t l_z l^{\omega-1} + \frac{\beta_t^2}{2}(\omega l_{zz} l^{\omega-1} + \omega(\omega-1) l_z^2 l^{\omega-2}) - \rho\beta_t\sigma_t\overline{\pi}_t\left(\gamma\omega e^{\int_t^T r_s ds} l_z l^{\omega-1}\right) + \kappa e^{-\gamma\widetilde{R_t}F e^{\int_t^T r_s ds}} g - \kappa l^{\omega}\} = 0$$

We consider,

$$\Phi(\overline{\pi}_t) = -\overline{\pi}_t(\mu_t - r_t)\gamma e^{\int_t^T r_s ds} l^{\omega} + \frac{\sigma_t^2\overline{\pi}_t^2}{2}\left(\gamma^2 e^{2\int_t^T r_s ds} l^{\omega}\right) + \omega\alpha_t l_z l^{\omega-1} + \frac{\beta_t^2}{2}(\omega l_{zz} l^{\omega-1} + \omega(\omega-1) l_z^2 l^{\omega-2}) - \rho\beta_t\sigma_t\overline{\pi}_t\left(\gamma\omega e^{\int_t^T r_s ds} l_z l^{\omega-1}\right) + \kappa e^{-\gamma\widetilde{R_t}F e^{\int_t^T r_s ds}} g - \kappa l^{\omega}$$

the first-order optimality condition on $\Phi(\overline{\pi}_t)$ is

$$\frac{d\Phi(\overline{\pi}_t)}{d\overline{\pi}_t} = 0 \Leftrightarrow -(\mu_t - r_t)\gamma e^{\int_t^T r_s ds} l^{\omega} + \sigma_t^2\overline{\pi}_t\gamma^2 e^{2\int_t^T r_s ds} l^{\omega} - \rho\beta_t\sigma_t\left(\gamma\omega e^{\int_t^T r_s ds} l_z l^{\omega-1}\right) = 0 \Leftrightarrow \sigma_t^2\overline{\pi}_t\gamma e^{\int_t^T r_s ds} = (\mu_t - r_t) + \rho\beta_t\sigma_t\omega\frac{l_z}{l}$$

we deduce,

$$\overline{\pi}_t^\star = \frac{1}{\gamma} e^{-\int_t^T r_s ds}\left(\frac{\mu_t - r_t}{\sigma_t^2} + \frac{\rho\beta_t\omega}{\sigma_t}\frac{l_z}{l}\right) \quad (49)$$

the previous PDE gives,

$$\omega l_t l^{\omega-1} - \overline{\pi}_t^{\star}(\mu_t - r_t)\gamma e^{\int_t^T r_s ds} l^{\omega} + \frac{\sigma_t^2 (\overline{\pi}_t^{\star})^2}{2}\left(\gamma^2 e^{2\int_t^T r_s ds} l^{\omega}\right) + \omega\alpha_t l_z l^{\omega-1} + \frac{\beta_t^2}{2}(\omega l_{zz} l^{\omega-1} + \omega(\omega-1) l_z^2 l^{\omega-2}) - \rho\beta_t\sigma_t\overline{\pi}_t^{\star}\left(\gamma\omega e^{\int_t^T r_s ds} l_z l^{\omega-1}\right) + \kappa e^{-\gamma\widetilde{R_t}F e^{\int_t^T r_s ds}} g - \kappa l^{\omega} = 0$$

and after the substitution $\overline{\pi}_t^{\star}$ by its value and reorganization of the terms, we finally obtain the equation (19) defined by:

$$l_t + \frac{\beta_t^2}{2} l_{zz} + \left[\alpha_t - \frac{\rho\beta_t(\mu_t - r_t)}{\sigma_t}\right] l_z + \left[\frac{(\mu_t - r_t)^2}{2\omega\sigma_t^2} + \frac{\kappa}{\omega}\right] l + \frac{\beta_t^2}{2}(-\rho^2\omega + \omega - 1)\frac{l_z^2}{l} + \frac{\kappa}{\omega} e^{-\gamma\widetilde{R_t}F e^{\int_t^T r_s ds}} \frac{g}{l^{\omega-1}} = 0.$$

This indicates the derivation of the nonlinear partial differential equation in l.

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